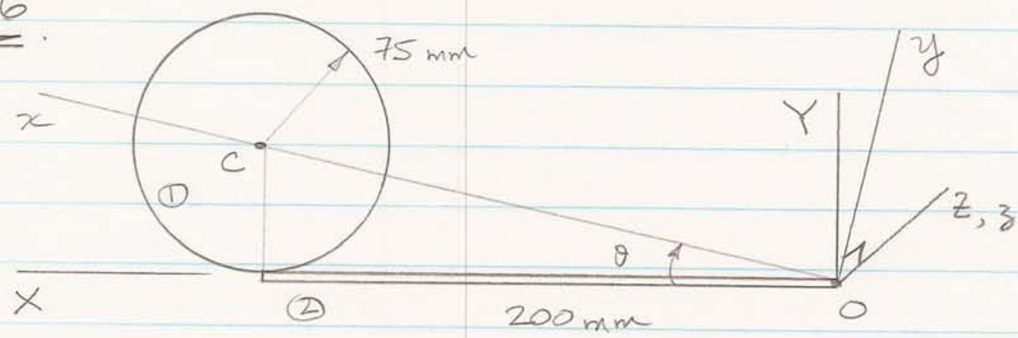
SD 553 EXTRA PROBLEMS © J. McPhee

5.6



$m_1 = 10$ kg
$m_2 = 2$ kg

- Use ∥-axis theorem to get $[I_o]$ in XY frame
- Use rotation transformation to xy frame

For the circle,

$$I_{XX_1} = \underbrace{\frac{1}{4} m_1 r^2}_{\text{appendix}} + \underbrace{m_1 (0.075)^2}_{\text{distance}^2}$$

$$= \frac{5}{4}(10)(0.075)^2 = 0.07031 \text{ kg}\cdot\text{m}^2$$

$$I_{YY_1} = \frac{1}{4} m_1 r^2 + m_1 (0.2)^2 = 0.4141$$

$$I_{ZZ_1} = \frac{1}{2} m_1 r^2 + m_1 (0.2^2 + 0.075^2) = 0.4844$$

Since XY is a plane of symmetry, $I_{XZ} = I_{YZ} = 0$.

$$I_{XY_1} = 0 + m_1(-0.075)(-0.2) = 0.15$$

For the rod,

$$I_{XX_2} = 0$$

$$I_{YY_2} = I_{ZZ_2} = \frac{1}{12} m_2 \ell^2 + m_2 (\ell/2)^2 = \frac{1}{3} m_2 \ell^2$$

$$= \frac{1}{3}(2)(0.2)^2 = 0.0267$$

$$I_{xy_2} = I_{yz_2} = I_{xz_2} = 0 \qquad \text{(principal axes)}$$

Combining,

$$I_{xx} = I_{xx_1} + I_{xx_2} = 0.07031 \quad \text{, et cetera.}$$

giving:
$$[I_o] = \begin{bmatrix} 0.07031 & -0.15 & 0 \\ -0.15 & 0.4407 & 0 \\ 0 & 0 & 0.5110 \end{bmatrix}$$

Now, [R] corresponds to a rotation of θ about $+z$:

$$[R] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \tan\theta = 75/200, \quad \theta = 20.56^\circ$$

Finally,

$$[I_o]' = [R][I_o][R]^T$$

$$= \begin{bmatrix} 0.01316 & 0.01433 & 0 \\ -0.1651 & 0.4654 & 0 \\ 0 & 0 & 0.5110 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ANS $\quad \therefore \quad [I_o]' = \begin{bmatrix} 0.0173 & 0.00876 & 0 \\ 0.00876 & 0.494 & 0 \\ 0 & 0 & 0.511 \end{bmatrix} \text{kg}\cdot\text{m}^2$

Note the relatively small magnitude of $I_{xy} = -0.00876$, indicating that we are close to the principal axes' orientation.

From Matlab,

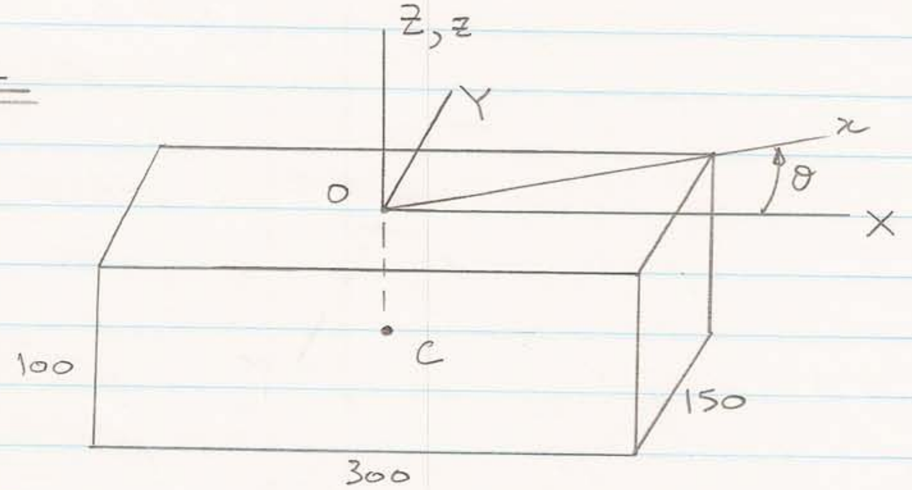
$I_1 = 0.0172$ $\quad \hat{e}_1 = -0.9998\,\hat{i} + 0.0184\,\hat{j}$

$I_2 = 0.4938$ $\quad \hat{e}_2 = -0.9998\,\hat{j} - 0.0184\,\hat{i}$

$I_3 = 0.5110$ $\quad \hat{e}_3 = \hat{k}$

corresponding to a rotation (about z) of 1.05° to the principal axes.

5.12



$m = 24$ kg
$a = 300$ mm
$b = 150$ mm
$c = 100$ mm

- use ||-axis theorem to get $[I_o]$ in XYZ components
- use rotation matrix $[R]$ to get $[I_o']$ in xyz "

Since XYZ are principal axes, $I_{XY} = I_{XZ} = I_{YZ} = 0$

$$I_{xx} = \underbrace{\frac{1}{12} m (b^2 + c^2)}_{\text{App.}} + m \underbrace{\left(\frac{c}{2} \right)^2}_{\text{distance}} = 0.125 \text{ kg}\cdot\text{m}^2$$

$$I_{yy} = \frac{1}{12} m(a^2+c^2) + m\left(\frac{c}{2}\right)^2 = 0.26 \text{ kg}\cdot\text{m}^2$$

$$I_{zz} = \frac{1}{12} m(a^2+b^2) = 0.225 \text{ kg}\cdot\text{m}^2$$

$$\therefore \quad [I_o] = \begin{bmatrix} 0.125 & 0 & 0 \\ 0 & 0.26 & 0 \\ 0 & 0 & 0.225 \end{bmatrix} \text{kg}\cdot\text{m}^2$$

Now, $[R]$ corresponds to a rotation of θ about z:

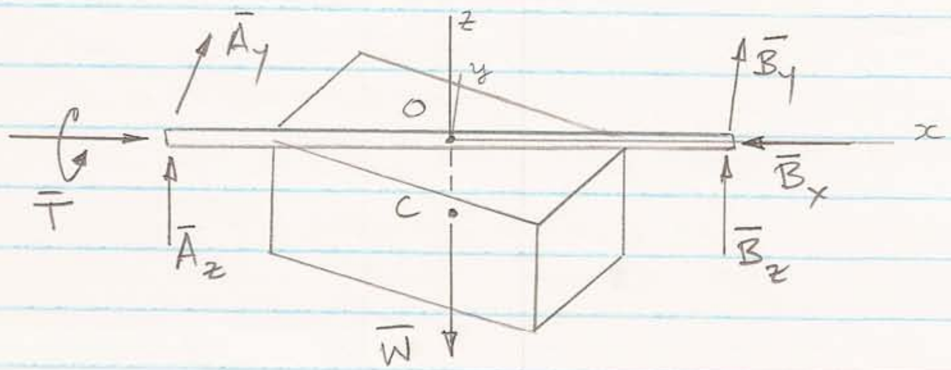
$$[R] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \tan\theta = 150/300 \quad \therefore \theta = 26.565^\circ$$

from which,

$$[I_o'] = [R][I_o][R]^T = \begin{bmatrix} 0.152 & 0.054 & 0 \\ 0.054 & 0.233 & 0 \\ 0 & 0 & 0.225 \end{bmatrix} \text{kg}\cdot\text{m}^2$$

ANS.

FBD of Block:



$\bar{T}$ = driving torque (for ω)

AO = 0.4 m

BO = 0.4 m

OC = 0.05 m

Note that no reaction moments develop at the "properly-aligned" joints, since the reaction forces are sufficient to prevent rotation about y and z.

Kinematics

$$\bar{\omega} = \omega \hat{i} \quad , \quad \dot{\bar{\omega}} = \dot{\omega} \hat{i} = 0$$

$$\bar{a}_c = OC\, \omega^2 \hat{k} = 0.05\, \omega^2 \hat{k} \quad \text{(centripetal only)}$$

Dynamics

$$\bar{F} = -B_x \hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z - W)\hat{k} = m\, \bar{a}_c$$

$$= 0.05\, m \omega^2 \hat{k}$$

$$\therefore \; B_x = 0 \quad ; \quad A_y + B_y = 0 \; , \quad A_y = -B_y$$

$$A_z + B_z = mg + 0.05\, m\omega^2$$

To get "dynamic" reactions, W is not included (it contributes to "static" reactions).

$$\therefore \; A_z + B_z = 0.05\, m\, \omega^2 = 1.2\, \omega^2$$

Now, $\{H_o\} = [I_o]\{\omega\}$ where $\{\omega\} = [\omega, 0, 0]^T$

$$\therefore \; \bar{H}_o = 0.152\, \omega\, \hat{i} + 0.054\, \omega\, \hat{j}$$

and $\bar{\omega} \times \bar{H}_o = 0.054\, \omega^2 \hat{k}$

$$\{\dot{H}_o\} = [I_o]\{\dot{\omega}\} = \{0\} \quad \text{since } \dot{\bar{\omega}} = 0$$

Note that we cannot use Euler equations since xyz are <u>not</u> principal axes.

$$\overline{M}_o = \overset{\circ}{\overline{H}}_o + \overline{\omega} \times \overline{H}_o \qquad (O \text{ is fixed})$$

$$= 0.054\,\omega^2 \hat{k}$$

$$\left.\begin{aligned} &= 0.4 A_z \hat{j} - 0.4 B_z \hat{j} + 0.4 B_y \hat{k} \\ &\quad -0.4 A_y \hat{k} \end{aligned}\right\} \text{moments about O due to reactions}$$

Note that $\overline{W}$ would create a moment about x (when $\overline{OC} \parallel \hat{k}$) that would be counterbalanced by $\overline{T}$ to maintain ω.

Equating components, $\quad 0.4 A_z - 0.4 B_z = 0$

$$\therefore \; A_z = B_z, \quad 2A_z = 1.2\,\omega^2, \quad A_z = 0.6\,\omega^2$$

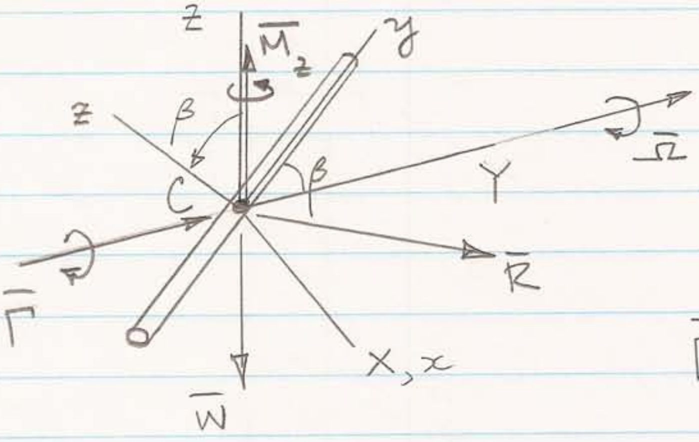
and, $\quad 0.4 B_y - 0.4 A_y = 0.054\,\omega^2$

$$\therefore \; 0.8 B_y = 0.054\,\omega^2 \qquad \therefore \; B_y = 0.0675\,\omega^2$$

Summaring the dynamic bearing reactions,

$$\boxed{\begin{aligned} \overline{A} &= \omega^2(-0.0675\,\hat{j} + 0.6\,\hat{k}) \quad (N) \\ \overline{B} &= \omega^2(0.0675\,\hat{j} + 0.6\,\hat{k}) \quad (N) \end{aligned}}$$

5.20 FBD of bar



Ω = constant

$\overline{R} = \overline{R}_x + \overline{R}_y + \overline{R}_z$ = pin reaction

$\overline{M}_z$ = pin moment

$\overline{\Gamma}$ = driving torque.

DOF = 1 (β), Unknown loads = 5 $(R_x, R_y, R_z, M_z, \Gamma)$

Kinematics: Let xyz be body-fixed axis such that $\hat{\imath} \parallel \hat{I}$ at this instant.

$$\overline{\omega} = \Omega \hat{J} + \dot{\beta}\hat{\imath} = \dot{\beta}\hat{\imath} + \Omega(\cos\beta\,\hat{\jmath} - \sin\beta\,\hat{k})$$

$$\therefore \quad \dot{\overline{\omega}} = \underbrace{\dot{\Omega}}_{0}\hat{J} + \ddot{\beta}\hat{\imath} + \dot{\beta}(\overline{\Omega} \times \hat{\imath})$$

$$= \ddot{\beta}\hat{\imath} - \dot{\beta}\Omega\hat{K} = \ddot{\beta}\hat{\imath} - \dot{\beta}\Omega(\sin\beta\,\hat{\jmath} + \cos\beta\,\hat{k})$$

$$\overline{a}_c = 0$$

Dynamics

$$\overline{F} = m\overline{a}_c = \overline{R} + \overline{W} = 0 \qquad \therefore \overline{R} = -\overline{W} = mg\hat{K}$$

For rotation, we can use Euler's equations for the xyz principal axes, where:

$$I_{xx} = I_{zz} = \frac{1}{12} mL^2 , \quad I_{yy} = 0 \qquad \text{(about c)}$$

$$\therefore \quad M_{c_x} = I_{xx}\alpha_x - (I_{yy} - I_{zz})\omega_y \omega_z$$

$$= \frac{1}{12} ml^2 \ddot{\beta} + \frac{1}{12} ml^2 \Omega^2 (\cos\beta)(-\sin\beta)$$

$$= 0 \text{ from the FBD}$$

$$\therefore \quad \boxed{\ddot{\beta} - \Omega^2 \cos\beta \sin\beta = 0}$$ is the ODE to be solved for $\beta(t)$.

$$M_{c_y} = 0 - (0)\omega_x \omega_z$$

$$= M_z \sin\beta + \Gamma \cos\beta \quad \text{from FBD.}$$

$$\therefore \quad M_z = -\Gamma \frac{\cos\beta}{\sin\beta}$$

$$M_{c_z} = \frac{1}{12} mL^2 (-\dot{\beta}\Omega\cos\beta) - \frac{1}{12} mL^2 (\dot{\beta})(\Omega\cos\beta)$$

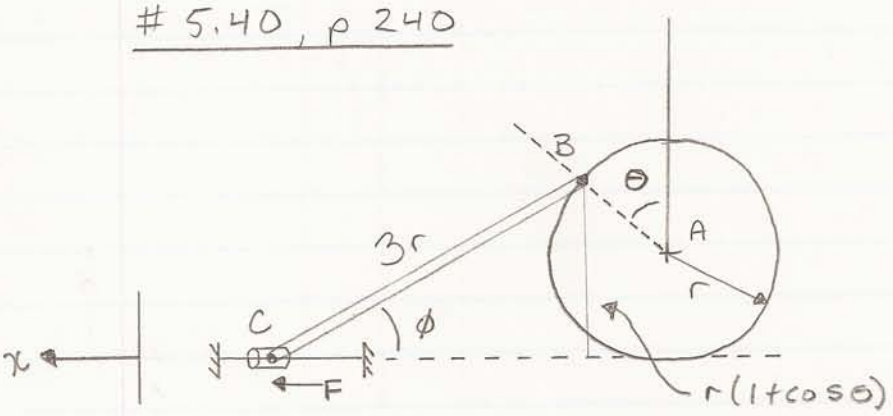
$$= -\frac{mL^2}{6} \Omega \dot{\beta} \cos\beta = M_z \cos\beta - \Gamma \sin\beta \text{, from FBD}$$

$$\therefore \quad -\frac{mL^2}{6}\Omega\dot{\beta}\cos\beta = -\Gamma\left(\frac{\cos^2\beta}{\sin\beta} + \sin\beta\right) = -\frac{\Gamma}{\sin\beta}$$

$$\therefore \quad \boxed{\Gamma = \frac{mL^2}{6} \Omega \dot{\beta} \sin\beta \cos\beta}$$

(Note that inertia transformation needed for XYZ components)

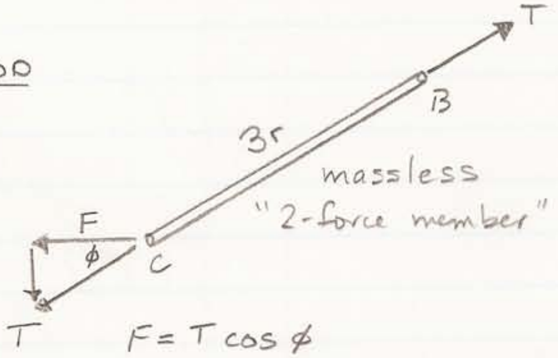
5.40, p 240



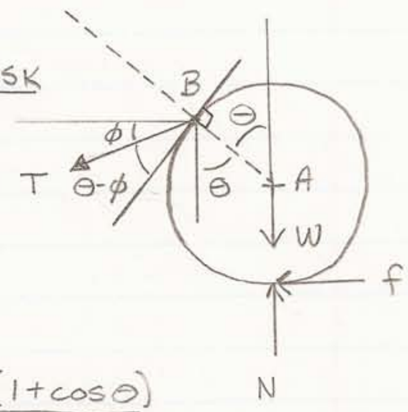
DISK ROLLS
WITHOUT SLIPPING
MASS m
RADIUS OF GYRATION K

CONSIDER A FREE BODY DIAGRAM OF THE DISK. AND ROD.

ROD



DISK



$F = T\cos\phi$

$T = \dfrac{F}{\cos\phi}$ where $\sin\phi = \dfrac{r(1+\cos\theta)}{3r}$

$I = mK^2$

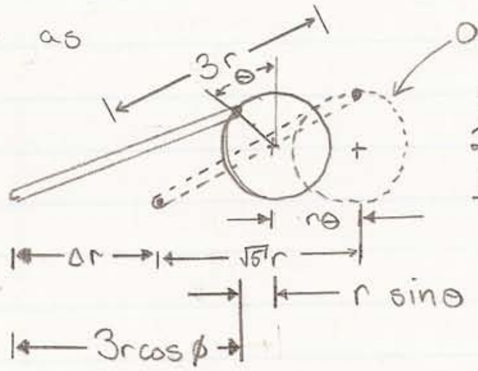
The above free body diagrams are not necessary to solve the problem if we apply Work-Energy directly at point C.

$$\cancelto{0}{T_1} + \int F \cdot dr_C = T_2$$

$$F\,\Delta r = \tfrac{1}{2} m v_A^2 + \tfrac{1}{2} I \omega_A^2$$

The reaction forces N and f do no work on the system.

FOR Δr AS



$\Delta r = 3r\cos\phi + r\sin\theta + r\theta - \sqrt{5}\,r$

Substituting for Δr and $\omega_A = \frac{V_A}{r}$ into Work-Energy

$$F(3r\cos\phi + r\sin\theta + r\theta - \sqrt{5}\,r) = \frac{1}{2} m V_A^2 + \frac{1}{2} I \frac{V_A^2}{r^2}$$

$$Fr(3\cos\phi + \sin\theta + \theta - \sqrt{5}) = V_A^2 \left(\frac{1}{2} m + \frac{1}{2r^2}(mK^2)\right)$$

$$Fr(3\cos\phi + \sin\theta + \theta - \sqrt{5}) = V_A^2 \frac{m}{2}\left(1 + \frac{K^2}{r^2}\right)$$

<u>ASIDE:</u> Recall $\sin\phi = \frac{r(1+\cos\theta)}{3r}$

$$\cos\phi = \frac{\sqrt{9r^2 - r^2(1+\cos\theta)^2}}{3r}$$

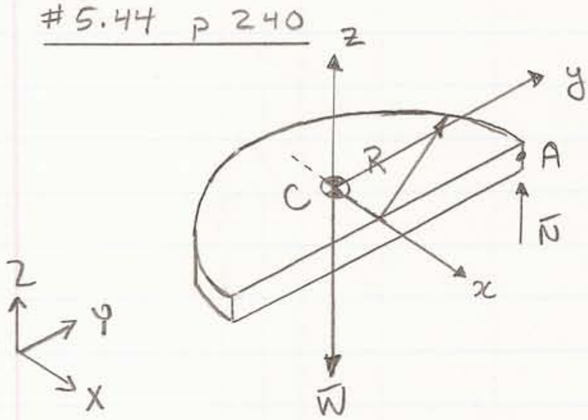
$$= \frac{r(8 - 2\cos\theta - \cos^2\theta)^{1/2}}{3r}$$

Then resubstituting for V_A

$$V_A^2 = \frac{2F}{m}\left(\frac{r^3}{r^2+K^2}\right)\left[(8 - 2\cos\theta - \cos^2\theta)^{1/2} + \sin\theta + \theta - \sqrt{5}\right]$$

Alternatively, we could have applied work energy at point A, using equations 5.98, and 5.99 (p. 216)

#5.44 p 240



We assume the plate is 'thin'

Define a local body fixed frame at the centre of mass of the disk.

Prior to collision disk is travelling at a velocity of $-v\hat{K}$

During the collision with the ledge, we expect force $\bar{N}$ to act upon the disk at point A. $\bar{N}$ will be large and act for a very short instant.

Writing the impulse momentum equation for the disk at the instant the collision occurs.

$$m\bar{v}_{c1} + \int_{\Delta t} \bar{F}\, dt = m\bar{v}_{c2}$$

$$m\bar{v}_{c1} + \int_{\Delta t} \bar{W}\, dt + \int_{\Delta t} \bar{N}\, dt = m\bar{v}_{c2}$$

where $\bar{v}_{c1}$ and $\bar{v}_{c2}$ are the velocities of the disk CM before and after the collision

$\int_{\Delta t} \bar{W}\, dt = 0 \rightarrow$ This is a non-impulse force since it is very small with respect to $\int_{\Delta t} \bar{N}\, dt$

For simplicity, we assume $\bar{N}$ is constant over the duration of the collision.

$$-mv\hat{K} + N\Delta t\,\hat{K} = m\bar{v}_{c2}$$

$$(N\Delta t - mv)\hat{K} = m\bar{v}_{c2}$$

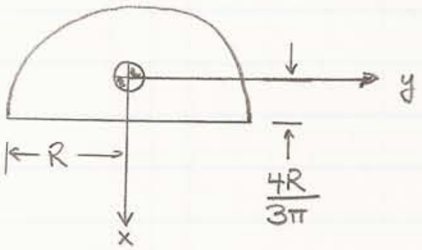
$\hat{K}$: $$v_{c2}^{z} = \frac{(N\Delta t - mv)}{m} \quad (1)$$

Writing the Angular impulse momentum equation at the instant of the collision.

$$\bar{H}_1 + \int_{\Delta t} \bar{M}\, dt + \bar{H}_2$$

$$\cancelto{0}{\bar{H}_1} + \int_{\Delta t} \bar{r}_{A/C} \times \bar{N} \, dt = \bar{H}_2$$

ASIDE → CENTRE OF MASS AND INERTIA PROPERTIES.



The easiest way to determine inertia properties for this shape is to examine the prederived cases for the Semicylinder (p.442) as the dimension L becomes small.

$$I_{xx} = \frac{1}{12} m \left(3R^2 + \cancelto{0}{L^2}\right) \qquad I_{yy} = \frac{9\pi^2 - 64}{36\pi^2} mR^2 + \frac{1}{12} m\cancelto{0}{L^2}$$

$$= \frac{mR^2}{4} \qquad\qquad = 0.06987 mR^2$$

$$I_{zz} = \frac{9\pi^2 - 32}{18\pi^2} mR^2 = 0.3199 mR^2$$

Alternatively, one could have modified ½ the prederived cases for a thin disk with the parallel axis theorem.

$$\bar{r}_{A/C} = \frac{4R}{3\pi}\hat{i} + R\hat{j} \qquad \bar{H}_2 = I_{xx}\omega_x \hat{i} + I_{yy}\omega_y \hat{j} + I_{zz}\omega_z \hat{k}$$

$\Omega_2 = (\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k})$ is angular velocity after the collision.

Substituting into angular impulse expression.

$$\left(-\frac{4R}{3\pi}\hat{j} + R\hat{i}\right) N \Delta t = I_{xx}\omega_x \hat{i} + I_{yy}\omega_y \hat{j} + I_{zz}\omega_z \hat{k}$$

Taking Components of this expression

$$\hat{i}: \quad RN\Delta t = I_{xx}\omega_x \quad (2)$$

$$\hat{j}: \quad -\frac{4R}{3\pi} N \Delta t = I_{yy}\omega_y \quad (3)$$

$$k: \quad 0 = I_{zz}\omega_z \quad (4)$$

FINDING THE VELOCITY OF POINT A

$$\bar{v}_A = \bar{v}_C + \bar{\Omega} \times \bar{r}_{C/A}$$

At the instant after the collision, we are told that $\bar{v}_{A_2} = v\hat{k}$

$$\bar{v}_{A_2} = \bar{v}_{C_2} + \bar{\Omega}_2 \times \bar{r}_{C/A}$$

$$v\hat{k} = \bar{v}_{C_2} + (\omega_x \hat{\imath} + \omega_y \hat{\jmath} + \omega_z \hat{k}) \times \left(\frac{4R}{3\pi}\hat{\imath} + R\hat{\jmath}\right)$$

$$v\hat{k} = \bar{v}_{C_2} + \omega_x R\hat{k} - \omega_y \frac{4R}{3\pi}\hat{k} + \omega_z \frac{4R}{3\pi}\hat{\jmath} - \omega_z \hat{\imath}$$

$$v\hat{k} = \bar{v}_{C_2} + (-\omega_z)\hat{\imath} + \left(\omega_z \frac{4R}{3\pi}\right)\hat{\jmath} + \left(\omega_x R - \omega_y \frac{4R}{3\pi}\right)\hat{k} \qquad (5)$$

At the instant right after the collision, $\hat{\imath} = \hat{I}$, $\hat{\jmath} = \hat{J}$, $\hat{k} = \hat{K}$. We have 7 equations in 7 unknowns $(v_{C2}^x, v_{C2}^y, v_{C2}^z, \omega_x, \omega_y, \omega_z, N\Delta t)$

Solving... From (4) $\omega_z = 0$

From (4), (1), (5) $\quad v_{C2}^x = 0$, $v_{C2}^y = 0$

FROM above, (2), (3), (5) $\quad N\Delta t = 0.2639\, mv$

FROM above, (2), (3) $\quad \omega_x = \dfrac{1.055 v}{R} \quad \omega_y = -\dfrac{1.601 v}{R}$

FROM above, (1) $\quad v_{C2}^z = -0.7361 v$

SUMMARIZING... At the instant after the collision, the velocity of the plate is

$$\bar{v}_C = -0.736 v \hat{k}$$
$$\bar{\Omega}_c = 1.055 \frac{v}{R}\hat{\imath} - 1.601 \frac{v}{R}\hat{\jmath}$$

Where the collision force is given by:

$$N = 0.264 \frac{mv}{\Delta t}$$

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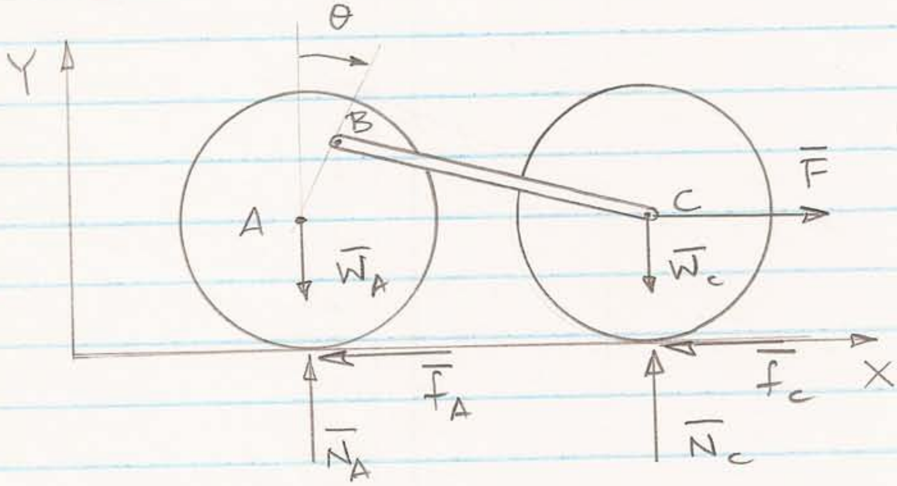
#6.25

Rolling w/o slipping.

$\therefore$ DOF = 1



use $q = \theta$

$m_A = m_c = m$

$r_A = r_c = R$

$m_{BC} = 0$

$BC = 3R$

$AB = \epsilon$

$\overline{f}_A, \overline{N}_A, \overline{f}_c, \overline{N}_c$ do no work for no slipping

$\overline{W}_A, \overline{W}_c$ do no work since $y_A = y_c =$ constant

$\therefore$ only $\overline{F}$ contributes to Q_θ:

$$\delta W = \overline{F} \cdot \delta \overline{r}_c = Q_\theta \, \delta\theta$$

Kinematics

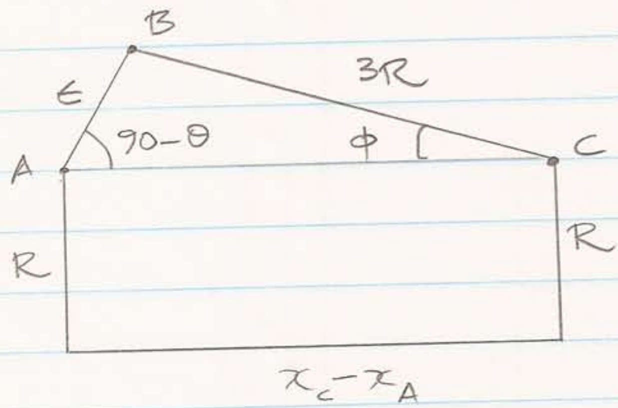
$$\overline{v}_A = R\dot{\theta}\,\hat{\imath} \quad \text{(no slipping)}, \quad \overline{\omega}_A = -\dot{\theta}\,\hat{k}$$

$$\overline{v}_B = \overline{v}_A + \overline{\omega}_A \times \overline{r}_{B/A} = \overline{v}_A - \dot{\theta}\,\hat{k} \times \epsilon(\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})$$

$$= \left(R\dot{\theta} + \epsilon\dot{\theta}\cos\theta\right)\hat{\imath} - \epsilon\dot{\theta}\sin\theta\,\hat{\jmath}$$

$$\overline{v}_c = \overline{v}_B + \overline{\omega}_{BC} \times \overline{r}_{c/B} = v_c\,\hat{\imath}$$

But we need a position analysis to get $\bar{r}_{c/B}$:



$$\frac{\sin\phi}{\epsilon} = \frac{\sin(90-\theta)}{3R} = \frac{\cos\theta}{3R}$$

$$\therefore \quad \sin\phi = \frac{\epsilon}{3R}\cos\theta \qquad (1)$$

$$\cos\phi = \sqrt{1-\sin^2\phi} \qquad (2)$$

$$\therefore \quad \bar{r}_{c/B} = 3R(\cos\phi\,\hat{\imath} - \sin\phi\,\hat{\jmath}), \quad \bar{\omega}_{Bc} = -\dot{\phi}\,\hat{k}$$

$$\therefore \quad v_c\,\hat{\imath} = \bar{v}_B - \dot{\phi}\,\hat{k} \times \bar{r}_{c/B}$$

$$= (R\dot{\theta} + \epsilon\dot{\theta}\cos\theta - 3R\dot{\phi}\sin\phi)\,\hat{\imath}$$
$$+ (-\epsilon\dot{\theta}\sin\theta - 3R\dot{\phi}\cos\phi)\,\hat{\jmath}$$

Equating components, $\quad 0 = -\epsilon\dot{\theta}\sin\theta - 3R\dot{\phi}\cos\phi$

i.e. $\quad \dot{\phi} = -\dfrac{\epsilon\dot{\theta}\sin\theta}{3R\cos\phi} \quad$ (differentiate (1) to confirm)

$$\therefore \quad v_c = R\dot{\theta} + \epsilon\dot{\theta}\cos\theta - \cancel{3R}\left(-\frac{\epsilon\dot{\theta}\sin\theta}{\cancel{3R}\cos\phi}\right)\sin\phi$$

$$= \dot{\theta}\left(R + \epsilon\cos\theta + \epsilon\sin\theta\,\frac{\sin\phi}{\cos\phi}\right) = f(\theta)\,\dot{\theta}$$

where $\sin\phi$, $\cos\phi$ are given by (1-2). We could also have obtained v_c by using trigonometry to get x_c, and differentiating.

For completeness, $\quad \omega_c = \dfrac{v_c}{R} = \dfrac{f \dot{\theta}}{R}$

Now,

$$\frac{d\bar{r}_c}{\cancel{dt}} = \bar{v}_c = f \frac{d\theta}{\cancel{dt}} \hat{\imath}$$

$\therefore$ using the "kinematical method",

$$\delta \bar{r}_c = f \, \delta\theta \, \hat{\imath}$$

$$\therefore \quad \delta W = \bar{F} \cdot \delta \bar{r}_c = F\hat{\imath} \cdot f \delta\theta \, \hat{\imath} = Ff \, \delta\theta = Q_\theta \, \delta\theta$$

$$\therefore \quad Q_\theta = Ff = F\left(R + \epsilon \cos\theta + \epsilon \sin\theta \frac{\sin\phi}{\cos\phi} \right)$$

Expanding f,

$$f = R + \epsilon \cos\theta + \frac{\epsilon \sin\theta}{\cancel{3R}} \frac{\epsilon \cos\theta}{\left(\dfrac{9R^2 - \epsilon^2 \cos^2\theta}{\cancel{9R^2}} \right)^{1/2}}$$

$$\therefore \quad f = R + \epsilon \cos\theta + \frac{\epsilon^2 \sin\theta \cos\theta}{\sqrt{9R^2 - \epsilon^2 \cos^2\theta}} \qquad (3)$$

Now to calculate T:

$$T = \frac{1}{2} m v_A^2 + \frac{1}{2} I_A \omega_A^2 + \frac{1}{2} m v_c^2 + \frac{1}{2} I_c \omega_c^2$$

$$= \frac{m}{2}\left[R^2 \dot{\theta}^2 + \frac{R^2}{2} \dot{\theta}^2 + f^2 \dot{\theta}^2 + \frac{\cancel{R^2}}{2} \frac{f^2}{\cancel{R^2}} \dot{\theta}^2 \right] \qquad \left(I = \frac{mR^2}{2} \right)$$

$$L = T - V = T = \frac{3}{4} m (R^2 + f^2) \dot{\theta}^2$$

$\therefore \quad \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}} \right) - \dfrac{\partial L}{\partial \theta} = Q_\theta = Ff \quad$ becomes:

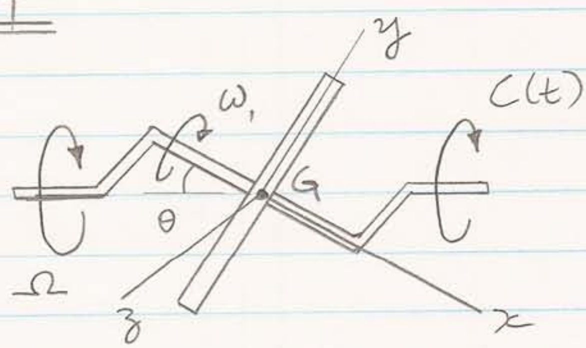
$$\frac{d}{dt}\left(\frac{3}{2}m(R^2+f^2)\dot{\theta}\right) - \left(\frac{3}{2}mf\dot{\theta}^2\frac{\partial f}{\partial \theta}\right) = Ff$$

$$\therefore \frac{3}{2}m(R^2+f^2)\ddot{\theta} + 3mf\dot{\theta}^2\frac{\partial f}{\partial \theta} - \frac{3}{2}mf\dot{\theta}^2\frac{\partial f}{\partial \theta} = Ff$$

or simply: $$\boxed{\frac{3}{2}m\left[(R^2+f^2)\ddot{\theta} + f\frac{\partial f}{\partial \theta}\dot{\theta}^2\right] = Ff}$$

where $\partial f/\partial \theta$ is obtained by differentiating (3).

#6-34



$\omega_1 = \omega_1(t) = \dot{\phi}$
(servomotor)

$\Omega = \Omega(t) = \dot{\psi}$
(torque $C(t)$)

Since ω_1 is a specified function of time, the system has only 1 DOF corresponding to $\Omega = \dot{\psi}$.

$\therefore$ let $q = \psi$

None of the constraint reactions, including the servomotor torque, does work during a virtual displacement $\delta\psi$.

$$\therefore \delta W = C\,\delta\psi = Q_\psi\,\delta\psi \qquad \therefore Q_\psi = C(t)$$

Since the center of gravity G is fixed, $V = 0$.

Finally,

$$T = \frac{1}{2} m \cancelto{0}{v_G^2} + \frac{1}{2}\{\omega\}^T [I_G]\{\omega\}$$

where: $\bar{\omega} = \bar{\Omega} + \bar{\omega}_1$

$$= \Omega(-\cos\theta\,\hat{i} - \sin\theta\,\hat{j}) - \omega_1\hat{j}$$

$$= (-\omega_1 - \Omega\cos\theta)\hat{i} - \Omega\sin\theta\,\hat{j}$$, valid for any ψ.

$$I_{xx} = \frac{1}{2} m R^2, \quad I_{yy} = I_{zz} = \frac{1}{4} m R^2$$

$$\therefore \; T = \frac{1}{2}\left[\frac{1}{2} m R^2 (-\omega_1 - \Omega\cos\theta)^2 + \frac{1}{4} m R^2 (-\Omega\sin\theta)^2\right]$$

$$= \frac{mR^2}{4}\left[\omega_1^2 + 2\omega_1\Omega\cos\theta + \Omega^2\cos^2\theta + \frac{\Omega^2\sin^2\theta}{2}\right]$$

$$L = T - V = T$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) - \cancelto{0}{\frac{\partial L}{\partial \psi}} = Q_\psi = C(t)$$

Thus, knowing $C(t)$, we could integrate this equation to get an impulse-momentum equation for ψ.

$$\frac{\partial L}{\partial \dot{\psi}} = p_\psi = \frac{\partial L}{\partial \Omega} = \frac{mR^2}{4}\left[2\omega_1\cos\theta + 2\Omega\cos^2\theta + \Omega\sin^2\theta\right]$$

i.e. $$p_\psi\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} C(t)\,dt$$

Since we are only asked for the ODE (not a solution), we differentiate to get:

$$\frac{mR^2}{4}\left[2\dot{\omega}_1\cos\theta + 2\dot{\Omega}\cos^2\theta + \dot{\Omega}\sin^2\theta\right] = C(t)$$

or

$$\boxed{\dot{\Omega}\left(2\cos^2\theta + \sin^2\theta\right) = \frac{4C}{mR^2} - 2\dot{\omega}_1\cos\theta}$$

which can be solved for $\Omega(t)$.

Note that we could also solve for the servomotor torque by:

① let ω_1 be uncontrolled ($f = 2$ DOF)

② let $q_1 = \psi$, $q_2 = \phi$

③ derive eqns. for q_1 and q_2

④ set $\omega_1 = \omega_1(t)$ and solve q_1 eqn. for $\psi(t)$

⑤ solve q_2 eqn. for servomotor torque.