Graph-Theoretic Modelling of Multibody Systems

by

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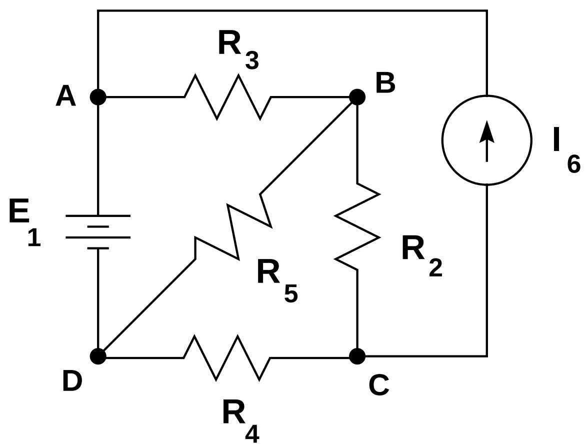
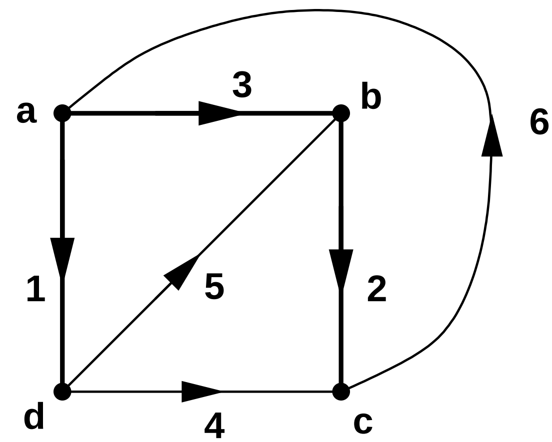
Part 1: Introduction

- Origins of linear graph theory
- Graph-theoretic modelling of physical systems
- Multibody system dynamics
- Summary of presentation

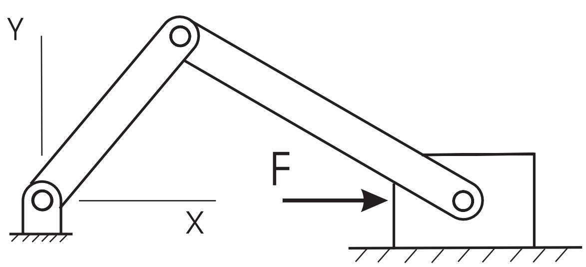
G-T Modelling of Physical Systems (2002):

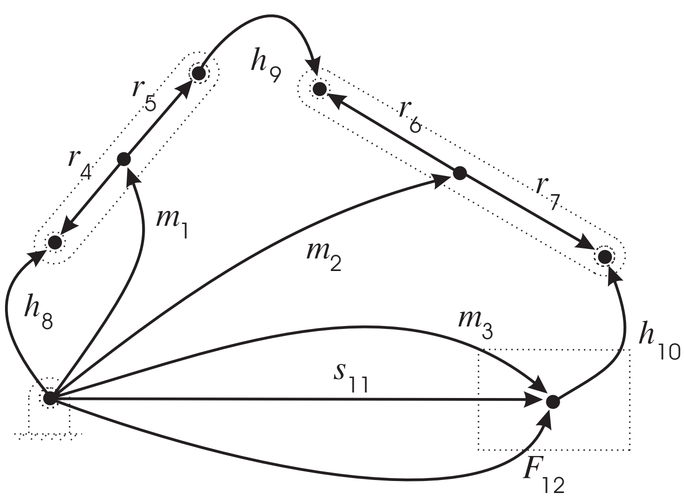




Through variable $\boldsymbol{\tau} = i$ (current)

Across variable $\boldsymbol{\alpha} = v$ (voltage)





Through variable $\boldsymbol{\tau} = \underline{F}$ or $\underline{T}$

Across variable $\boldsymbol{\alpha} = \underline{r}\,, \underline{\dot{r}}\,, \underline{\ddot{r}}$ or $\underline{\theta}\,, \underline{\omega}\,, \underline{\dot{\omega}}$

Features of Graph-Theoretic Models:

- explicit representation of topology
- well-developed mathematical theory

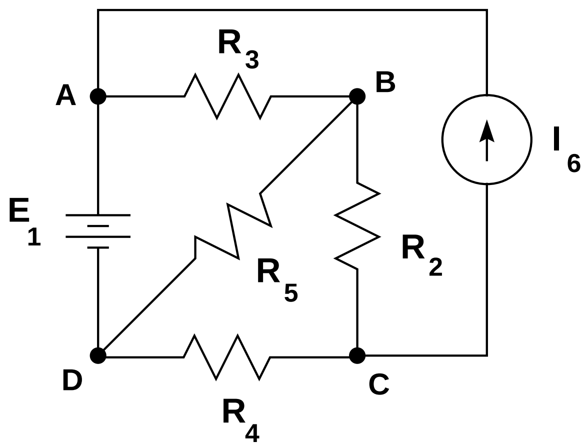
- separation of topological and constitutive equations
- coordinates defined by tree selection
- provides a unifying approach to multibody dynamics
- systematic formulation procedures (vectorial and analytical mechanics)
- efficient computer implementations

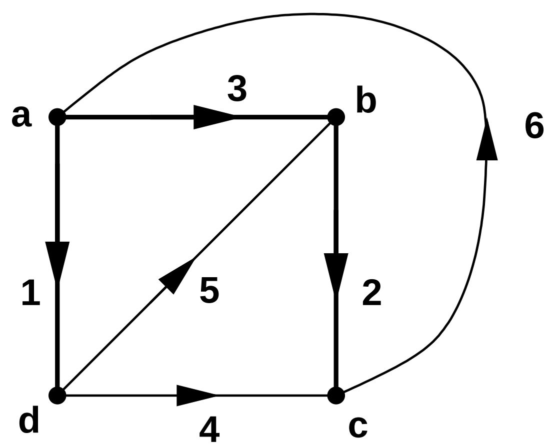
- equally applicable to electrical, mechanical, and other domains
- applicable to multi-domain systems (e.g. "mechatronic systems")
- special topologies (e.g. parallel robots) can be exploited
- subsystem models can be developed, nested, and re-used

Part 2: Basics of Graph-Theoretic Modelling (GTM)

- Linear graph = nodes + edges
- Physical variables and constitutive equations
- Incidence matrix
- Cutset and circuit equations
- Example formulation

- **Linear Graph**: A collection of e edges which intersect only at v vertices, or nodes. Each edge is said to be **incident** upon the nodes at its ends.





- **Tree**: A subgraph of a connected graph that:

  1. is connected
  2. contains all v nodes
  3. has no circuits

  In the tree, there are $b = v - 1$ edges known as **branches**. In the cotree, there are $c = e - v + 1$ edges known as **chords**.

- **Fundamental Circuit**: A circuit containing one chord and a unique set of branches. There is one f-circuit for each chord.

- **Cutset**: A subgraph that, when removed, divides a connected graph into exactly two parts, and no subset of this subgraph has this property.

- **F-Cutset**: A cutset consisting of one branch and a unique set of chords. There is one fundamental cutset for each branch.

- **Directed Graph**: A linear graph with oriented edges. A directed graph can be used to represent a physical system; edges correspond to physical components, while nodes represent the points of connection.

- **Through Variable $\boldsymbol{\tau}$**: A physical variable measured by an instrument placed *in series* with the corresponding element.

- **Across Variable $\boldsymbol{\alpha}$**: A physical variable measured by an instrument placed *in parallel* with the corresponding element.

Physical Domain	$\boldsymbol{\tau}$	$\boldsymbol{\alpha}$
Electrical	Current i	Voltage v
Hydraulic	Flow rate q	Pressure P
1-D Mechanical	Force F	Displacement x

Examples of Through and Across Variables

- **Terminal Equation**: The linear or nonlinear equations that model the constitutive properties of individual components, e.g.

  Linear electrical resistor: $v = R\,i$

  Pipe (hydraulic resistance): $P = R\,q^k$

  There are e terminal equations in terms of the $2e$ unknown through and across variables.

- **Incidence Matrix IM**: Matrix representation of directed graph.

$$\mathbf{IM}_{jk} = \left\{ \begin{array}{c} 0 \\ +1 \\ -1 \end{array} \right\} \text{ if edge } k \text{ is } \left\{ \begin{array}{c} \text{not incident on} \\ \text{incident on and away from} \\ \text{incident on and towards} \end{array} \right\} \text{ node } j$$

For the previous linear graph of an electrical circuit,

$$\mathbf{IM} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccccc} E_1 & R_2 & R_3 & R_4 & R_5 & I_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

- **Vertex Postulate**: The sum of through variables at any node of a linear graph must equal zero when due account is taken of the orientation of edges incident upon that node:

$$\mathbf{IM}\,\boldsymbol{\tau} = \mathbf{0}$$

For the electrical network example, one gets:

$$i_1 + i_3 - i_6 = 0$$

$$i_2 - i_3 - i_5 = 0$$

$$-i_2 - i_4 + i_6 = 0$$

$$-i_1 + i_4 + i_5 = 0$$

Note that the rank of $\mathbf{IM}$ is $b = e - 1$.

- **Cutset Equations:** Delete any one row of the singular incidence matrix to get the **reduced incidence matrix** $\mathbf{A}$:

$$\mathbf{A}\,\boldsymbol{\tau} = \mathbf{0}$$

The node corresponding to the deleted row is the **datum node**.

- **F-Cutset Equations:** Apply Gauss-Jordan reduction to $\mathbf{A}$ to get the f-cutset matrix $\mathbf{A_f}$:

$$\mathbf{A_f}\,\boldsymbol{\tau} = \mathbf{0}$$

where:

$$\mathbf{A_f} = \left[\mathbf{1}_b\,\mathbf{A}_c\right]$$

For the electrical example, taking d as the datum node,

$$\mathbf{A_f} = \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{bmatrix}$$

Alternatively, from a visual inspection of the graph,

$$\mathbf{A_f}_{\,jk} = \left\{\begin{matrix} 0 \\ -1 \\ +1 \end{matrix}\right\} \text{ if edge } k \text{ is } \left\{\begin{matrix} \text{not in the cutset for} \\ \text{in the cutset and against} \\ \text{in the cutset and with} \end{matrix}\right\} \text{ branch } j$$

- **Chord Transformation Equations:**

$$\boldsymbol{\tau}_b = -\mathbf{A}_c\,\boldsymbol{\tau}_c$$

which can be used to eliminate the b secondary variables $\boldsymbol{\tau}_b$.

- **Circuit Postulate:** The sum of across variables around any circuit of a graph must equal zero when due account is taken of the direction of edges in the circuit.

- **F-Circuit Equations:**

$$\mathbf{B_f}\ \boldsymbol{\alpha} = \mathbf{0}$$

where the **f-circuit matrix** $\mathbf{B_f}$ is constructed by:

$$\mathbf{B_f}_{\,jk} = \left\{ \begin{array}{c} 0 \\ -1 \\ +1 \end{array} \right\} \text{ if edge } k \text{ is } \left\{ \begin{array}{c} \text{not in the circuit for} \\ \text{in the circuit and against} \\ \text{in the circuit and with} \end{array} \right\} \text{ chord } j$$

which can be written in the partitioned form:

$$\mathbf{B_f} = \left[\mathbf{B}_{\,b}\, \mathbf{1}_{\,c}\right]$$

For the electrical example,

$$\mathbf{B_f} = \begin{bmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

- **Branch Transformation Equations:**

$$\boldsymbol{\alpha}_c = -\mathbf{B}_{\,b}\, \boldsymbol{\alpha}_b$$

which can be used to eliminate the c secondary variables $\boldsymbol{\alpha}_c$.

- **Principle of Orthogonality:**

$$\mathbf{A_f}\ \mathbf{B_f}^T = \mathbf{0}$$

which can be used to obtain $\mathbf{B}_b$ directly from $\mathbf{A}_c$:

$$\mathbf{B}_b = -\mathbf{A}_c^T$$

- **Algorithm for F-Cutset and F-Circuit equations:**

1. create incidence matrix $\mathbf{IM}$ for given system topology, with left-most columns corresponding to selected tree
2. delete row corresponding to datum node to get $\mathbf{A}$
3. apply Gauss-Jordan reduction to get $\mathbf{A_f}$ and chord transformations
4. use Principle of Orthogonality to get $\mathbf{B_f}$ and branch transforms

Together, the f-cutset and f-circuit equations (or branch and chord transformations) comprise a set of $b + c = e$ linear topological equations in terms of $2e$ through and across variables.

Formulation of System Equations

- **Nodal Formulation:** The e terminal equations are supplemented by the b cutset and e nodal transformation equations:

$$\boldsymbol{\alpha} = \mathbf{A}^T \boldsymbol{\alpha}_n$$

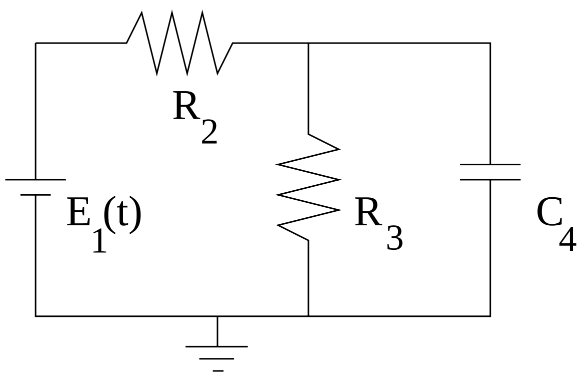
  which can be solved simultaneously for the $2e + b$ through, across, and nodal variables. No tree is specified.

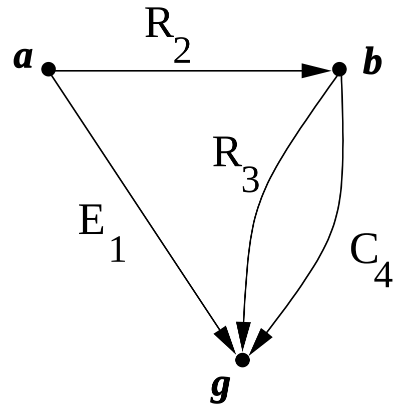
- **Branch-Chord Formulation:** Substitute the e branch and chord transformation equations into the terminal equations, resulting in e equations in e primary variables $\boldsymbol{\alpha}_b$ and $\boldsymbol{\tau}_c$.

- **Hybrid Formulation:** The number of simultaneous equations is further reduced by exploiting the form of the terminal equations, e.g. through drivers and selected into the cotree and across drivers are selected into the tree.

- **Tableau Methods:** Terminal equations with standard forms are assembled with cutset and circuit equations using a procedure analogous to the finite element method.

- **Example:**





$$\mathbf{IM} = \begin{array}{c} \\ a \\ b \\ g \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & R_2 & R_3 & C_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

Selecting g as the ground node,

$$\mathbf{A} = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & R_2 & R_3 & C_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \end{array}$$

- **Nodal Formulation**:

$$\boldsymbol{\alpha} = \mathbf{A}^T \boldsymbol{\alpha}_n$$

where the nodal voltages are:

$$\boldsymbol{\alpha}_n = [v_a, v_b]^T$$

In addition,

$$\mathbf{A}\boldsymbol{\tau} = \mathbf{0} \quad , \ \boldsymbol{\tau} = [i_1, i_2, i_3, i_4]^T$$

plus the terminal equations, assumed to be of the linear form:

$$\begin{aligned} v_1 &= E_1(t) \\ v_2 &= R_2 i_2 \\ v_3 &= R_3 i_3 \\ i_4 &= C_4 \dot{v}_4 \end{aligned}$$

subject to the initial condition $v_4(0) = v_{40}$.

Result is 10 linear differential-algebraic equations in terms of 8 through and across variables, and 2 nodal variables.

- **Branch-Chord Formulation**:

Selecting E_1 and C_4 as branches,

$$\mathbf{A_f} = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & C_4 & R_3 & R_2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \end{array} = \begin{bmatrix} \mathbf{1}_b \ \mathbf{A}_c \end{bmatrix}$$

Chord transformation equations:

$$\begin{Bmatrix} i_1 \\ i_4 \end{Bmatrix} = -\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{Bmatrix} i_3 \\ i_2 \end{Bmatrix}$$

Branch transformation equations:

$$\begin{Bmatrix} v_3 \\ v_2 \end{Bmatrix} = -\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_4 \end{Bmatrix}$$

Substituting into the terminal equations,

$$\mathcal{H}\left(\{v_b\}, \{i_c\}, t\right) = \begin{Bmatrix} v_1 - E_1(t) \\ (v_1 - v_4) - R_2 i_2 \\ v_4 - R_3 i_3 \\ (i_2 - i_3) - C_4 \dot{v}_4 \end{Bmatrix} = 0$$

Result is a set of 4 linear DAEs in terms of the 2 branch voltages and 2 chord currents.

- **Further Reductions:**

Exploiting the linear nature of the resistor elements,

$$\begin{aligned} i_2 &= \frac{E_1(t) - v_4}{R_2} \\ i_3 &= \frac{v_4}{R_3} \end{aligned}$$

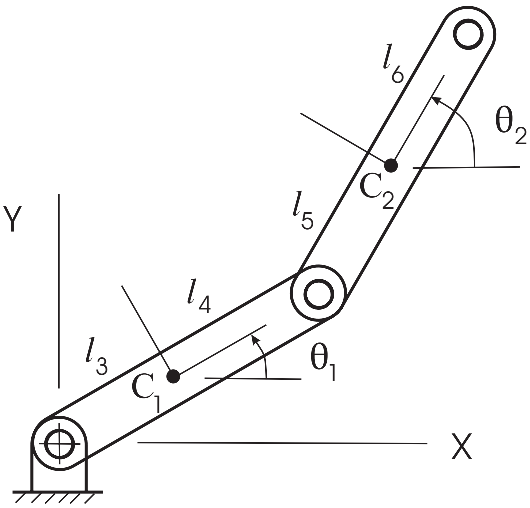
which are substituted into the last terminal equation to get:

$$C_4 \dot{v}_4 + \left(\frac{1}{R_3} + \frac{1}{R_2}\right) v_4 = \frac{E_1(t)}{R_2}$$

Part 3: Kinematics of Multibody Systems

- Coordinates and constraints in multibody system dynamics
- Linear graph representation
- Terminal equations
- Tree and coordinate selections
- Projected circuit equations
- Examples

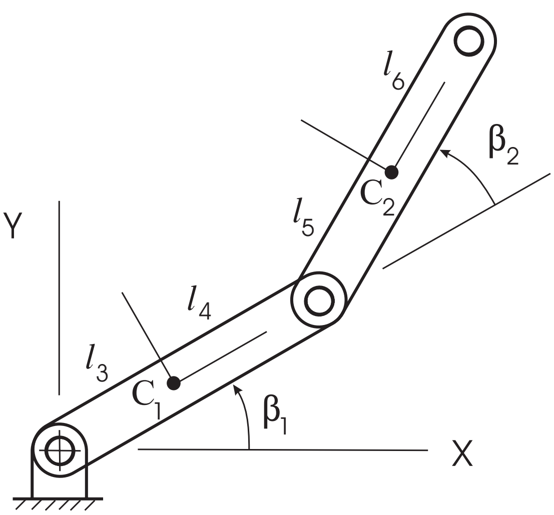
Absolute Coordinates $\mathbf{q_a}$



$$\mathbf{q_a} = [x_1, y_1, \theta_1, x_2, y_2, \theta_2]^T$$

- simple index notation to account for topology
- large systems of equations

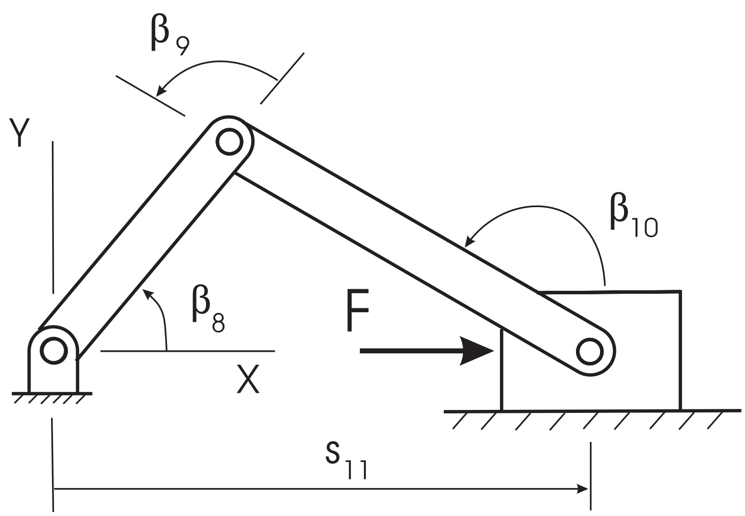
Joint Coordinates $\mathbf{q}_\beta$



$$\mathbf{q}_\beta = [\beta_1, \beta_2]^T$$

- topological accounting required

- independent for open-loop systems
- dependent for systems with closed kinematic chains:



For a system with f DOF modelled by n coordinates, we seek $m = n - f$ kinematic constraint equations, assumed to be of the holonomic form:

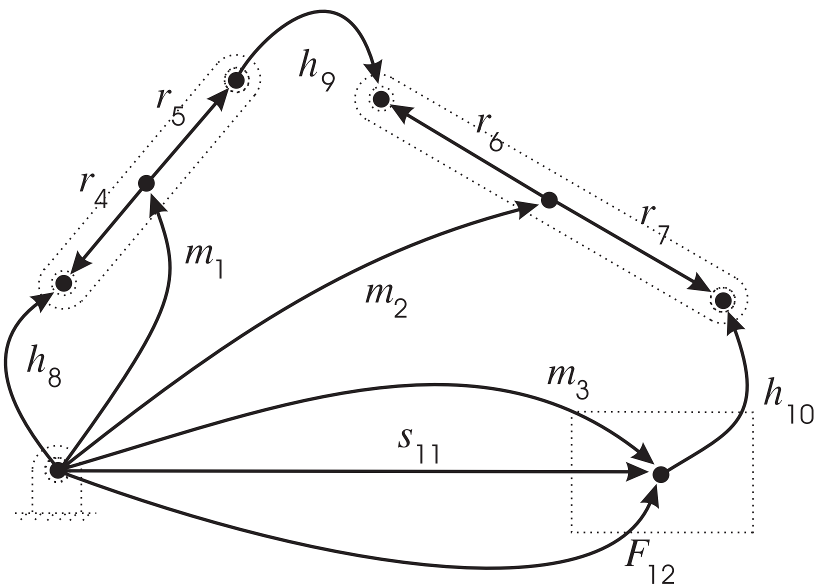
$$\mathbf{\Phi}(\mathbf{q}, t) = 0$$

with time derivatives:

$$\begin{aligned} \mathbf{\Phi}_{\mathbf{q}}\,\dot{\mathbf{q}} &= -\mathbf{\Phi}_{t} \equiv \boldsymbol{\nu} \\ \mathbf{\Phi}_{\mathbf{q}}\,\ddot{\mathbf{q}} &= -(\mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}})_{\mathbf{q}}\dot{\mathbf{q}} - 2\mathbf{\Phi}_{\mathbf{q}t}\dot{\mathbf{q}} - \mathbf{\Phi}_{tt} \equiv \boldsymbol{\gamma} \end{aligned}$$

A graph-theoretic approach will systematically generate these equations in absolute coordinates, or joint coordinates, or alternative sets that may be better suited to a given problem.

- **Linear graph representation:**



- nodes represent position and orientation of body-fixed reference frames
- edges represent transformations corresponding to physical components:
  - m represents the motion of an unconstrained rigid body relative to an inertial frame (node)
  - r, an "arm element", represents a constant kinematic transformation from the primary body frame (node) to a secondary frame on the same body
  - h represents a revolute (pin) joint
  - s represents a prismatic (slider) joint
  - F represents an external force
- Graphs of Wittenburg and Freudenstein are obtained by collapsing each body to a single node, and deleting all edges except the joints

- **Variables and Terminal Equations**

Note that we have separate variables for translation (T) and rotation (R):

$$\boldsymbol{\alpha}_T = \underline{r} \text{ or } \underline{v} \text{ or } \underline{a} \qquad \boldsymbol{\alpha}_R = \mathbf{R}(\theta) \text{ or } \underline{\omega} \text{ or } \underline{\dot{\omega}}$$

$$\boldsymbol{\tau}_T = \underline{F} \qquad \boldsymbol{\tau}_R = \underline{T}$$

Rigid Bodies:

$$\underline{r}_m = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \qquad \mathbf{R}_m = \mathbf{R}_m(\theta_m)$$

$$\underline{v}_m = \underline{\dot{r}}_m \qquad \underline{\omega}_m = \underline{\omega}_m(\theta_m, \dot{\theta}_m)$$

Rigid Arms:

$$\underline{r}_r = \underline{r}_r(\theta_m) \qquad \mathbf{R}_r = \mathbf{R}_r^0 = \text{constant}$$

$$\underline{v}_r = \underline{\omega}_m \times \underline{r}_r \qquad \underline{\omega}_r = 0$$

Ideal Joints:

$\mathcal{M}$ is the Motion Space spanned by the relative motions allowed by the joint, defined here by a set of unit vectors.

$\mathcal{F}$ is the Reaction Space spanned by the reaction loads arising in the joint, also defined by a set of unit vectors.

$$\mathcal{M}_T \cap \mathcal{F}_T = \emptyset \qquad \mathcal{M}_R \cap \mathcal{F}_R = \emptyset$$

$$\mathcal{M}_T \cup \mathcal{F}_T = \mathcal{R}^3 \qquad \mathcal{M}_R \cup \mathcal{F}_R = \mathcal{R}^3$$

Revolute Joint (joint axis $\hat{u}$, orthogonal axes $\hat{n}_1, \hat{n}_2$):

$\mathcal{M}_T = \emptyset$

$\mathcal{F}_T = (\hat{u}, \hat{n}_1, \hat{n}_2)$ or $(\hat{\imath}, \hat{\jmath}, \hat{k})$

$\underline{r}_h = r_h^0 \hat{u}$

$\underline{v}_h = 0$

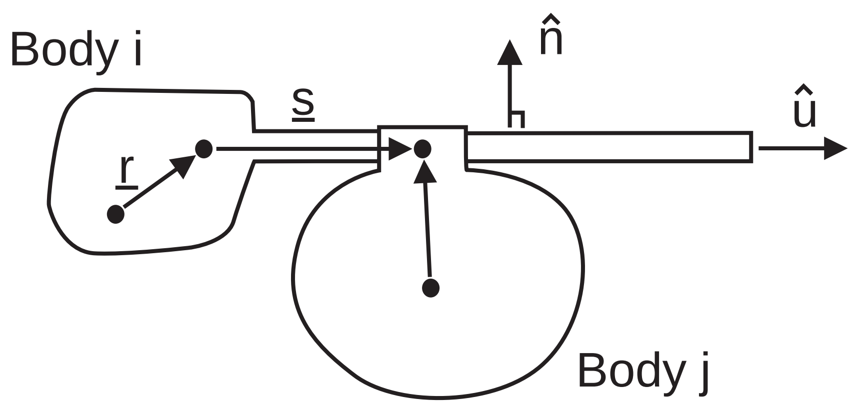
$\mathcal{M}_R = \hat{u}$

$\mathcal{F}_R = (\hat{n}_1, \hat{n}_2)$

$\mathbf{R}_h = \mathbf{R}_h(\beta, \hat{u})$

$\underline{\omega}_h = \dot{\beta}\hat{u}$

Prismatic Joint (joint axis $\hat{u}$, orthogonal axes $\hat{n}_1, \hat{n}_2$):



$\mathcal{M}_T = \hat{u}$

$\mathcal{F}_T = (\hat{n}_1, \hat{n}_2)$

$\underline{r}_s = s\hat{u}$

$\underline{v}_s = \dot{s}\hat{u} + \underline{\omega}_m \times \underline{r}_s$

$\mathcal{M}_R = \emptyset$

$\mathcal{F}_R = (\hat{u}, \hat{n}_1, \hat{n}_2)$ or $(\hat{\imath}, \hat{\jmath}, \hat{k})$

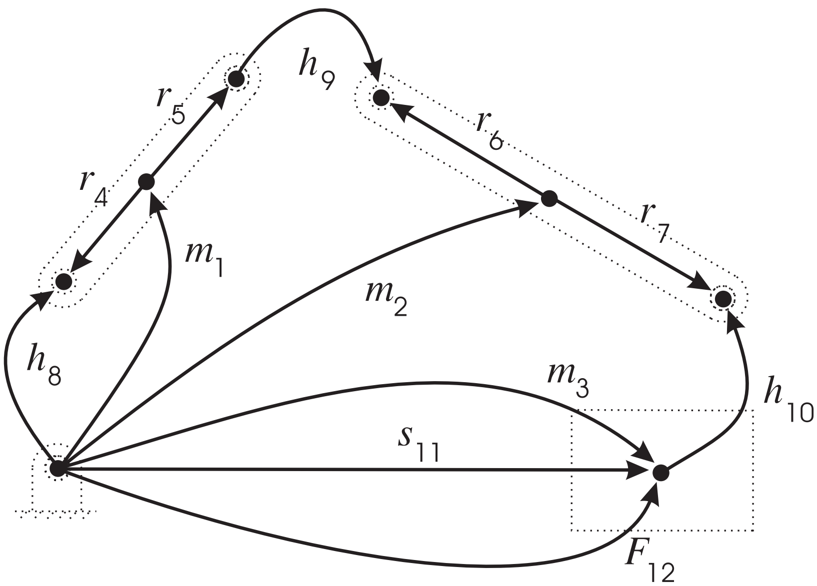
$\mathbf{R}_s = \mathbf{R}_s^0$

$\underline{\omega}_s = 0$

Other modelling components:

- Motion (across) drivers
- Other joints (spherical, universal, cylindrical, planar, weld, etc.)
- Dynamic elements (springs, dampers, forces, torques, etc.)

- **Tree and coordinate selection:**



Selection of a tree determines the "branch coordinates" appearing in the final equations of motion:

- r elements are always selected into tree, thereby reducing the number of unknown branch coordinates
- Tree $= [m_1, m_2, m_3]$ gives the $n = 9$ absolute coordinates $\mathbf{q_a}$
- Tree $= [h_8, h_{10}, s_{11}]$ gives $n = 3$ joint coordinates $\mathbf{q}_\beta = [\beta_8, \beta_{10}, s_{11}]^T$
- Tree $= [m_1, h_{10}, h_9]$ gives the $n = 5$ branch coordinates $\mathbf{q} = [x_1, y_1, \theta_1, \beta_{10}, \beta_9]^T$

One can choose separate trees for rotational and translational equations, in a way that minimizes the number of coordinates (and equations):

- T-Tree $= [h_8, h_9, h_{10}]$, R-Tree $= [h_8, h_{10}, s_{11}]$ gives the $n = 2$ branch coordinates $\mathbf{q} = [\beta_8, \beta_{10}]^T$

Tree selection priority for 2-dimensional systems:

Translational Tree

1. Rigid arms and motion drivers
2. Revolute joints
3. Prismatic joints
4. Rigid bodies
5. Force elements

Rotational Tree

1. Rigid arms and motion drivers
2. Prismatic joints
3. Rigid bodies
4. Revolute joints
5. Torque elements

Note that the use of body rotations (or "absolute angular coordinates", [Huston]) leads to less complex equations than the use of joint angles.

Tree selection priority for 3-dimensional systems:

Translational Tree

1. Rigid arms, motion drivers
2. Spherical, revolute, universal joints
3. Cylindrical, prismatic joints
4. Planar joints
5. Rigid bodies
6. Force elements

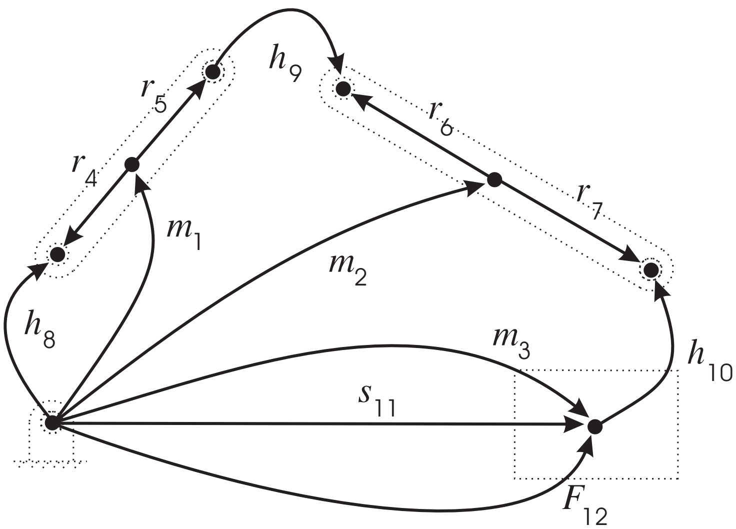
Rotational Tree

1. Rigid arms, motion drivers
2. Prismatic joints
3. Cylindrical, revolute joints
4. Rigid bodies
5. Spherical joints
6. Torque elements

A graph-theoretic formulation will generate equations in absolute, joint, or other coordinates. For any tree selection, the kinematic equations are obtained from the circuit equations.

- **Circuit equations and branch transformations:**

Once a tree is selected, one can automatically generate the circuit equations (and branch transformations) corresponding to kinematic loop closure.



For a single Tree $= [h_8, h_{10}, s_{11}]$ with $\mathbf{q} = [\beta_8, \beta_{10}, s_{11}]^T$,

$$\underline{r}_2 = \underline{r}_{11} + \underline{r}_{10} - \underline{r}_7$$

$$\underline{v}_2 = \underline{v}_{11} + \underline{v}_{10} - \underline{v}_7$$

$$\underline{\omega}_2 = \underline{\omega}_{11} + \underline{\omega}_{10} - \underline{\omega}_7$$

$$\underline{\theta}_2 = \underline{\theta}_{11} + \underline{\theta}_{10} - \underline{\theta}_7 \quad \text{for planar systems}$$

$$\mathbf{R}_2 = \mathbf{R}_{11}\mathbf{R}_{10}\mathbf{R}_7^T \quad \text{for spatial systems}$$

which allows all kinematic variables to be expressed in terms of $\mathbf{q}$.

The m kinematic constraint equations are obtained *by projecting the branch transformation for each cotree joint onto its reaction space* $\mathcal{F}$.

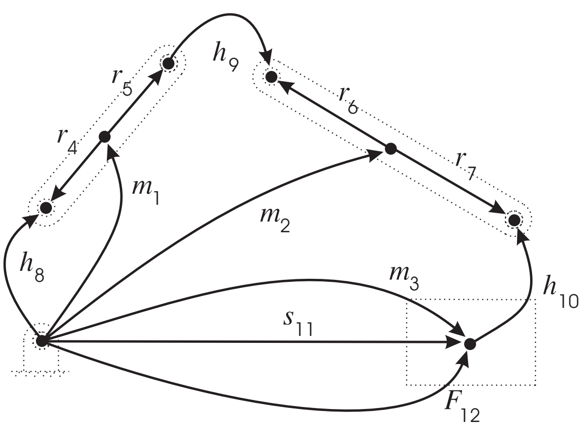
For a single Tree $= [h_8, h_{10}, s_{11}]$ with $\mathbf{q} = [\beta_8, \beta_{10}, s_{11}]^T$,

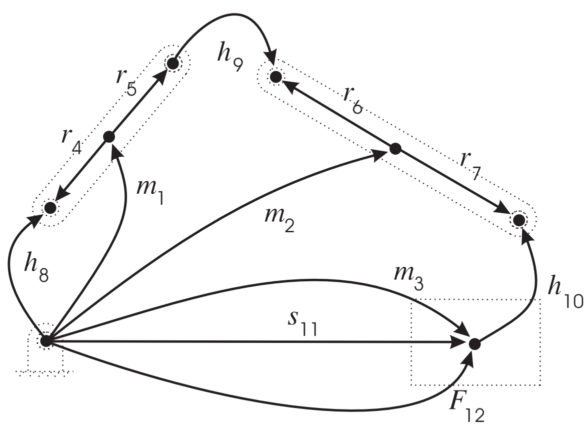
$$\begin{aligned}\mathbf{\Phi} &= \underline{r}_{\,9} \cdot \hat{\imath}\,, \hat{\jmath} \\ &= (\underline{r}_{\,6} - \underline{r}_{\,7} + \underline{r}_{\,10} + \underline{r}_{\,11} - \underline{r}_{\,8} + \underline{r}_{\,4} - \underline{r}_{\,5}) \cdot \hat{\imath}\,, \hat{\jmath}\end{aligned}$$

Substituting terminal equations and evaluating,

$$\mathbf{\Phi} = \left\{ \begin{array}{c} L_{67}\cos\beta_{10} + s_{11} - L_{45}\cos\beta_8 \\ L_{67}\sin\beta_{10} - L_{45}\sin\beta_8 \end{array} \right\} = \mathbf{0}$$

where $L_{45} = L_4 + L_5$, etc., and $L_i = |\underline{r}_{\,i}|$.





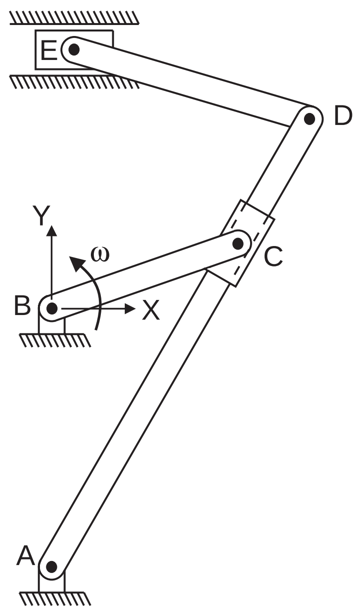
For a T-Tree $= [h_8, h_9, h_{10}]$, R-Tree $= [h_8, h_{10}, s_{11}]$, and $\mathbf{q} = [\beta_8, \beta_{10}]^T$,

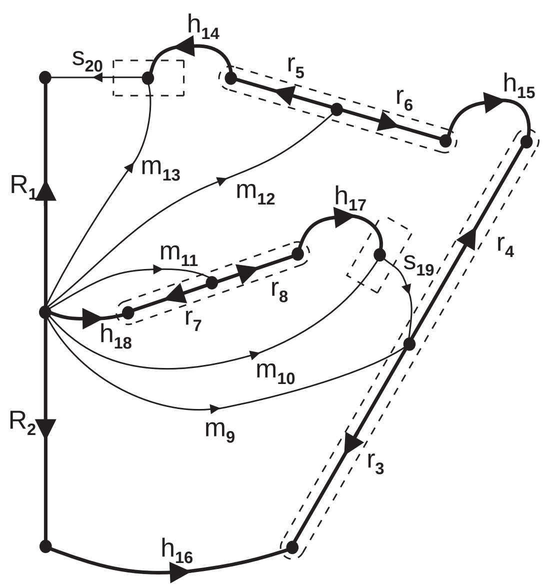
$$\begin{aligned}\mathbf{\Phi} &= \underline{r}_{\,11} \cdot \hat{\jmath} \\ &= (\underline{r}_{\,8} - \underline{r}_{\,4} + \underline{r}_{\,5} + \underline{r}_{\,9} - \underline{r}_{\,6} + \underline{r}_{\,7} - \underline{r}_{\,10}) \cdot \hat{\jmath}\end{aligned}$$

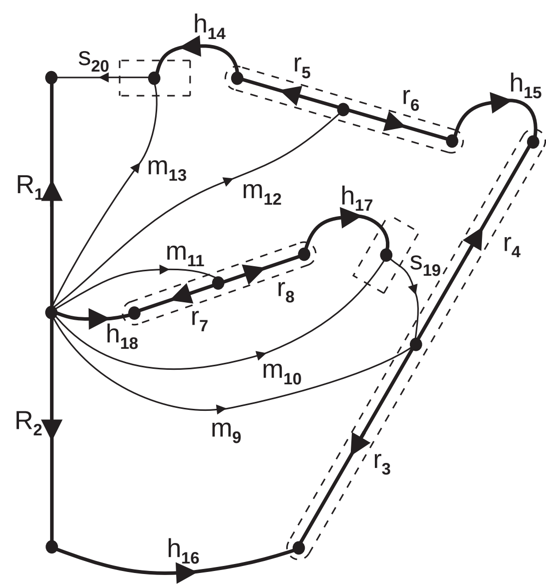
Substituting terminal equations and evaluating,

$$\mathbf{\Phi} = L_{45}\sin\beta_8 - L_{67}\sin\beta_{10} = \mathbf{0}$$

- Kinematics of Quick-Return Mechanism [Haug, 1989]:







For a T-Tree = $[h_{14-18}]$ and a R-Tree = $[m_9, m_{11}, m_{12}, s_{19-20}]$:

$$\mathbf{q} = [\theta_9, \theta_{11}, \theta_{12}]^T$$

Each prismatic joint in the T-tree contributes one constraint equation:

$$\mathbf{\Phi} = \left\{ \begin{array}{c} \underline{r}_{19} \cdot \hat{n}_{19} \\ \underline{r}_{20} \cdot \hat{n}_{20} \end{array} \right\} = \mathbf{0}$$

where the branch transformation equations are:

$$\begin{aligned} \underline{r}_{19} &= \underline{r}_2 + \underline{r}_{16} - \underline{r}_3 - \underline{r}_{17} - \underline{r}_8 + \underline{r}_7 - \underline{r}_{18} \\ \underline{r}_{20} &= \underline{r}_1 - \underline{r}_2 - \underline{r}_{16} + \underline{r}_3 - \underline{r}_4 + \underline{r}_{15} + \underline{r}_6 - \underline{r}_5 - \underline{r}_{14} \end{aligned}$$

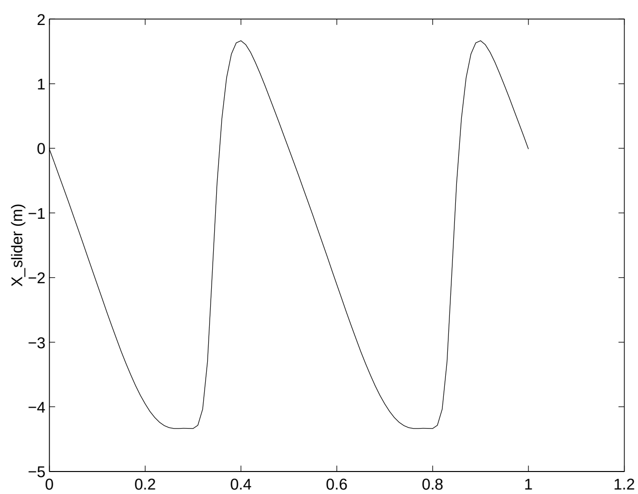
Substituting the terminal equations (e.g. $\hat{n}_{19} = \hat{n}_{19}(\theta_9)$, $\hat{n}_{20} = -\hat{\jmath}$), and evaluating:

$$\mathbf{\Phi} = \left\{ \begin{array}{c} L_{78} \sin\theta_9 \cos\theta_{11} - L_{78} \cos\theta_9 \sin\theta_{11} - L_2 \cos\theta_9 \\ -L_1 - L_2 + L_{34} \sin\theta_9 + L_{56} \cos\theta_{12} \end{array} \right\} = \mathbf{0}$$

which are easily solved for θ_9 and θ_{12}, given the prescribed motion for $\theta_{11} = 0.4356 + 4\pi t$. For comparison, one would need to solve 4 equations in joint coordinates, or 14 equations in absolute coordinates.

Projecting the branch transformation for s_{20} onto its joint axis $\hat{u}_{20} = -\hat{\imath}$ gives an explicit expression for the slider displacement:

$$s_{20} = L_{34} \cos\theta_9 - L_{56} \sin\theta_{12}$$



Part 4: Dynamics of Multibody Systems

- Governing equations in augmented form
- Terminal equations
- Projected cutset equations
- Principle of virtual work
- Embedding formulation

If less than f motion drivers are prescribed, then the determination of the time response requires the n dynamic equations:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda} = \mathbf{F}$$

which can be solved simultaneously with the previous m constraint equations

$$\mathbf{\Phi}(\mathbf{q}, t) = 0$$

for the n coordinates $\mathbf{q}$ and m Lagrange multipliers $\boldsymbol{\lambda}$.

The differential-algebraic equations resulting from the above "augmented formulation" may be solved using a wide variety of numerical methods (DAE solvers, generalized coordinate partitioning, Baumgarte's method, etc).

The dynamic equations can be obtained using either:

- cutset equations in terms of forces and torques (vectorial mechanics)
- the principle of virtual work (analytical mechanics)

- **Variables and Terminal Equations**

Recall the separate variables for translation (T) and rotation (R):

$$\boldsymbol{\alpha}_T = \underline{r} \text{ or } \underline{v} \text{ or } \underline{a} \qquad \boldsymbol{\alpha}_R = \mathbf{R}(\theta) \text{ or } \underline{\omega} \text{ or } \underline{\dot{\omega}}$$

$$\boldsymbol{\tau}_T = \underline{F} \qquad \boldsymbol{\tau}_R = \underline{T}$$

Revolute Joint (joint axis $\hat{u}$, orthogonal axes $\hat{n}_1, \hat{n}_2$):

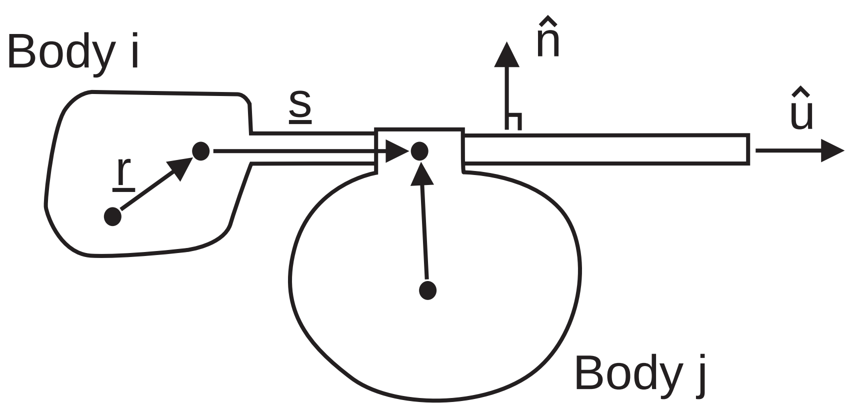
$$\mathcal{M}_T = \emptyset \qquad \mathcal{M}_R = \hat{u}$$

$$\mathcal{F}_T = (\hat{u}, \hat{n}_1, \hat{n}_2) \text{ or } (\hat{\imath}, \hat{\jmath}, \hat{k}) \qquad \mathcal{F}_R = (\hat{n}_1, \hat{n}_2)$$

$$\delta\underline{r}_h = 0 \qquad \delta\underline{\boldsymbol{\theta}}_h = \delta\beta\, \hat{u}$$

$$\underline{F}_h = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k} \qquad \underline{T}_h = T_1\hat{n}_1 + T_2\hat{n}_2$$

Prismatic Joint (joint axis $\hat{u}$, orthogonal axes $\hat{n}_1, \hat{n}_2$):



$$\mathcal{M}_T = \hat{u} \qquad \mathcal{M}_R = \emptyset$$

$$\mathcal{F}_T = (\hat{n}_1, \hat{n}_2) \qquad \mathcal{F}_R = (\hat{u}, \hat{n}_1, \hat{n}_2) \text{ or } (\hat{\imath}, \hat{\jmath}, \hat{k})$$

$$\delta\underline{r}_s = \delta s\, \hat{u} \qquad \delta\underline{\boldsymbol{\theta}}_s = 0$$

$$\underline{F}_s = F_1\hat{n}_1 + F_2\hat{n}_2 \qquad \underline{T}_s = T_x\hat{\imath} + T_y\hat{\jmath} + T_z\hat{k}$$

Note that $\delta W = \underline{F} \cdot \delta\underline{r} + \underline{T} \cdot \delta\underline{\theta} = 0$ for passive ideal joints.

Rigid Bodies:

$$\underline{r}_m = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \qquad \mathbf{R}_m = \mathbf{R}_m(\theta_m)$$
$$\underline{v}_m = \dot{\underline{r}}_m \qquad \underline{\omega}_m = \underline{\omega}_m(\theta_m, \dot{\theta}_m)$$
$$\mathcal{M}_T = \hat{\imath}, \hat{\jmath}, \hat{k} \qquad \mathcal{M}_R = \hat{\imath}, \hat{\jmath}, \hat{k}$$
$$\underline{F}_m = -m\underline{a}_m \qquad \underline{T}_m = -\underline{\underline{I}}_m \cdot \dot{\underline{\omega}}_m - \underline{\omega}_m \times \underline{\underline{I}}_m \cdot \underline{\omega}_m - \Sigma\, \underline{r} \times \underline{F}$$

where $\underline{\underline{I}}_m$ is the inertia dyadic for the body, about the center of mass, and the summation accounts for forces acting at other nodes on the body. For rigid arms, $\mathcal{M}_T = \mathcal{M}_R = \emptyset$.

Translational Spring-Damper:

$$\mathcal{M}_T = \hat{\imath}, \hat{\jmath}, \hat{k} \qquad \mathcal{M}_R = \hat{\imath}, \hat{\jmath}, \hat{k}$$
$$\underline{F}_{kd} = -[k(r - r_0) + d(\underline{v} \cdot \hat{r})]\hat{r} \qquad \underline{T}_{kd} = 0$$

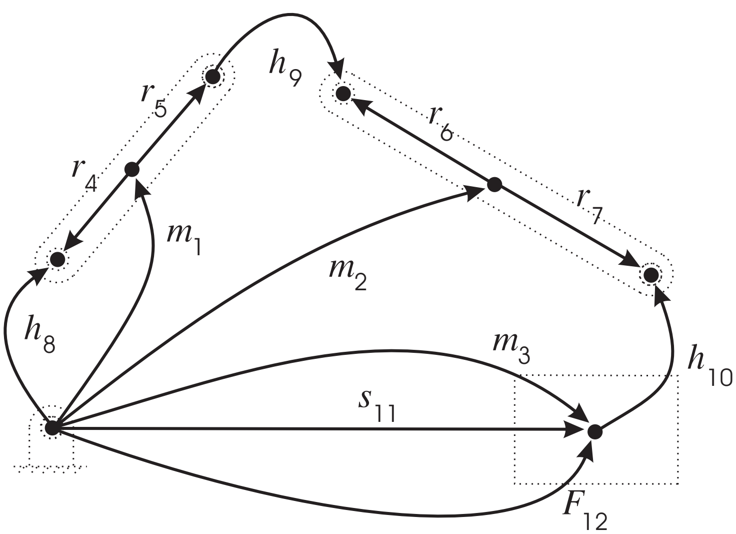
where k is the spring constant, $r = |\underline{r}|$ is the instantaneous length, r_0 is its undeformed length, d is the damping coefficient, and the unit vector $\hat{r} = \frac{\underline{r}}{r}$.

Other Components:

- motion drivers
- applied forces and torques (between any 2 nodes, any direction)
- rotational springs and dampers
- flexible bodies and flexible arms

- **Cutset equations and chord transformations:**

Once any tree is selected, one can automatically generate the cutset equations and chord transformations corresponding to dynamic equilibrium of individual or multiple bodies.



For a single Tree $= [h_8, h_{10}, s_{11}]$ with $\mathbf{q} = [\beta_8, \beta_{10}, s_{11}]^T$, the cutset equation for revolute joint h_8 is:

$$\underline{T}_8 + \underline{T}_1 - \underline{T}_9 = 0$$

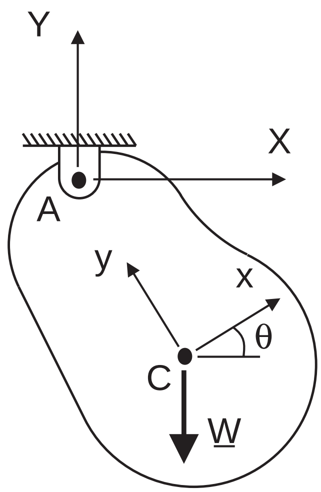
which represents a balance of torques acting on the crank. For s_{11},

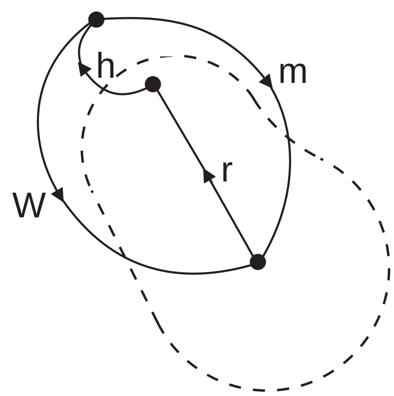
$$\underline{F}_{11} + \underline{F}_{12} + \underline{F}_2 + \underline{F}_3 + \underline{F}_9 = 0$$

which represents a balance of forces (including inertial) on the slider plus the connecting rod.

The n dynamic equations are obtained *by projecting the cutset equation for each component onto its motion space $\mathcal{M}$.*

- **Simple pendulum in absolute coordinates:**





Select a single Tree $= [r, m]$ to get $\mathbf{q} = [x_m, y_m, \theta_m]^T$.

Projecting the mass cutset equations onto its motion spaces,

$$(\underline{F}_{\,m} - \underline{F}_{\,h} + \underline{F}_{\,W} = 0) \cdot \hat{\imath}\,, \hat{\jmath}$$
$$(\underline{T}_{\,m} - \underline{T}_{\,h} = 0) \cdot \hat{k}$$

or, after substitution of the corresponding terminal equations,

$$-m\,\ddot{x}_m - F_x = 0$$
$$-m\,\ddot{y}_m - F_y - W = 0$$
$$-I\,\ddot{\theta}_m - (\underline{r}_{\,r} \times \underline{F}_{\,r}) \cdot \hat{k} \;= 0$$

where $\underline{F}_{\,r} = \underline{F}_{\,h}$ from the chord transformation for r, and the reaction force $\underline{F}_{\,h} = F_x\hat{\imath} + F_y\hat{\jmath}$. Substituting and assembling,

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda} = \mathbf{F}$$

where $\mathbf{M} = diag(m, m, I)$, $\boldsymbol{\lambda} = [F_x, F_y]^T$, $\mathbf{F} = [0, -W, 0]^T$, and:

$$\mathbf{\Phi}_{\mathbf{q}} = \begin{bmatrix} 1 & 0 & -r\cos\theta_m \\ 0 & 1 & -r\sin\theta_m \end{bmatrix}$$

which can be checked by projecting the circuit equation for h onto its reaction space:

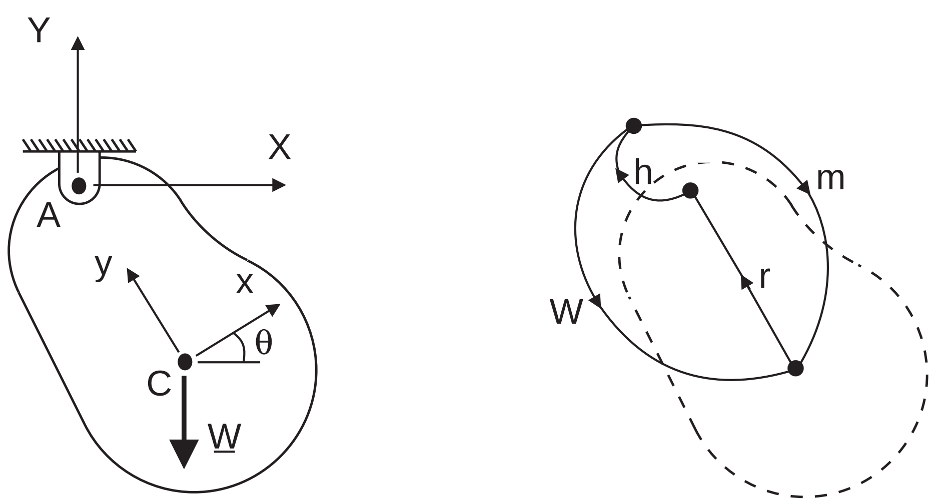
$$(\underline{r}_h + \underline{r}_m + \underline{r}_r = 0) \cdot \hat{\imath}\,, \hat{\jmath}$$

or, after substitution of the corresponding terminal equations,

$$\mathbf{\Phi} = \left\{ \begin{matrix} x_m - r\sin\theta_m \\ y_m + r\cos\theta_m \end{matrix} \right\} = \mathbf{0}$$

The use of absolute coordinates gives 5 DAEs to solve for $\mathbf{q}$ and $\boldsymbol{\lambda}$.

- **Simple pendulum in branch coordinates:**



Select a T-Tree $= [r, h]$ and a R-Tree $= [r, m]$ to get $\mathbf{q} = \theta_m$.

Projecting the cutset equations onto their motion spaces gives only

$$(\underline{T}_m - \underline{T}_h = 0) \cdot \hat{k}$$

or, after substitution of the corresponding terminal equations,

$$-I\ddot{\theta}_m - (\underline{r}_r \times \underline{F}_r) \cdot \hat{k} = 0$$

Using the chord transformation

$$\underline{F}_r = \underline{F}_m + \underline{F}_W = -m\,\underline{a}_m + \underline{W}$$

and the branch transformation

$$\underline{a}_m = -\underline{a}_r - \underline{a}_h = -\underline{\dot{\omega}}_m \times \underline{r}_r + \omega_m^2\,\underline{r}_r$$

results in the single ODE of motion:

$$\left(I + mr^2\right)\ddot{\theta}_m = -Wr\sin\theta_m$$

One can always get a minimum number of ODEs for open-loop systems, using joint or branch coordinates.

- **Virtual work formulation:**

We can formulate the augmented form of the dynamic equations by simply summing the contributions of n_T applied torques, n_F applied forces, n_R rigid bodies, and n_f flexible bodies to the system virtual work:

$$\delta W = \sum_{n_T} \mathbf{T}^T \delta\boldsymbol{\theta} + \sum_{n_F} \mathbf{F}^T \delta\mathbf{r} - \sum_{n_R} \left[m\mathbf{a}^T \delta\mathbf{r} + \left(\mathbf{I}\dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\mathbf{I}\boldsymbol{\omega}\right)^T \delta\boldsymbol{\theta} \right] + \sum_{n_f} \delta W_f$$

where $\mathbf{I}$ is the inertia matrix, $\tilde{\boldsymbol{\omega}}$ is the skew-symmetric matrix of the angular velocity $\boldsymbol{\omega}$, and the virtual work of a flexible body is:

$$\delta W_f = -\int_V \delta\boldsymbol{\epsilon}^T \boldsymbol{\sigma}\, dV + \int_V \delta\mathbf{r}^T \left(\mathbf{f}_b - \gamma\, \ddot{\mathbf{r}}\right) dV$$

where $\delta\boldsymbol{\epsilon}$ are varied strain components, $\boldsymbol{\sigma}$ are the corresponding stress components, $\mathbf{f}_b$ is the body force, and γ is the mass density.

Note that the cutset equations are not required. Using the branch transformations to express all quantities in terms of the branch coordinates $\mathbf{q}$,

$$\delta W = \delta\mathbf{q}^T \left[\mathbf{M}\ddot{\mathbf{q}} - \mathbf{F}\right] = 0$$

by the Principle of Virtual Work. The virtual displacements are kinematically admissible if and only if they satisfy the variation of the constraints:

$$\delta\boldsymbol{\Phi} = \boldsymbol{\Phi}_{\mathbf{q}} \delta\mathbf{q} = 0$$

We can use the Lagrange multiplier method to treat $\mathbf{q}$ as independent:

$$\delta W = \delta\mathbf{q}^T \left[\mathbf{M}\ddot{\mathbf{q}} + \boldsymbol{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda} - \mathbf{F}\right] = 0$$

resulting in the desired augmented form of the dynamic equations:

$$\mathbf{M}\ddot{\mathbf{q}} + \boldsymbol{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda} = \mathbf{F}$$

- **Embedding formulation:**

Symbolically partitioning the variation of the constraint equations into m dependent and f independent virtual displacements $\delta\mathbf{q}_d$ and $\delta\mathbf{q}_i$:

$$\delta\mathbf{\Phi} = \mathbf{\Phi}_{\mathbf{q}_d}\delta\mathbf{q}_d + \mathbf{\Phi}_{\mathbf{q}_i}\delta\mathbf{q}_i = \mathbf{0}$$

where $\mathbf{\Phi}_{\mathbf{q}_i} = \partial\mathbf{\Phi}/\partial\mathbf{q}_i$ and $\mathbf{\Phi}_{\mathbf{q}_d} = \partial\mathbf{\Phi}/\partial\mathbf{q}_d$ is $m \times m$ and non-singular. Solving,

$$\delta\mathbf{q}_d = -\mathbf{\Phi}_{\mathbf{q}_d}^{-1}\mathbf{\Phi}_{\mathbf{q}_i}\delta\mathbf{q}_i = \mathbf{J}\,\delta\mathbf{q}_i$$

Thus, the transformation from $\delta\mathbf{q}$ to independent $\delta\mathbf{q}$ is

$$\delta\mathbf{q} = \begin{bmatrix} \mathbf{J} \\ \mathbf{1} \end{bmatrix} \delta\mathbf{q}_i = \mathbf{G}\,\delta\mathbf{q}_i$$

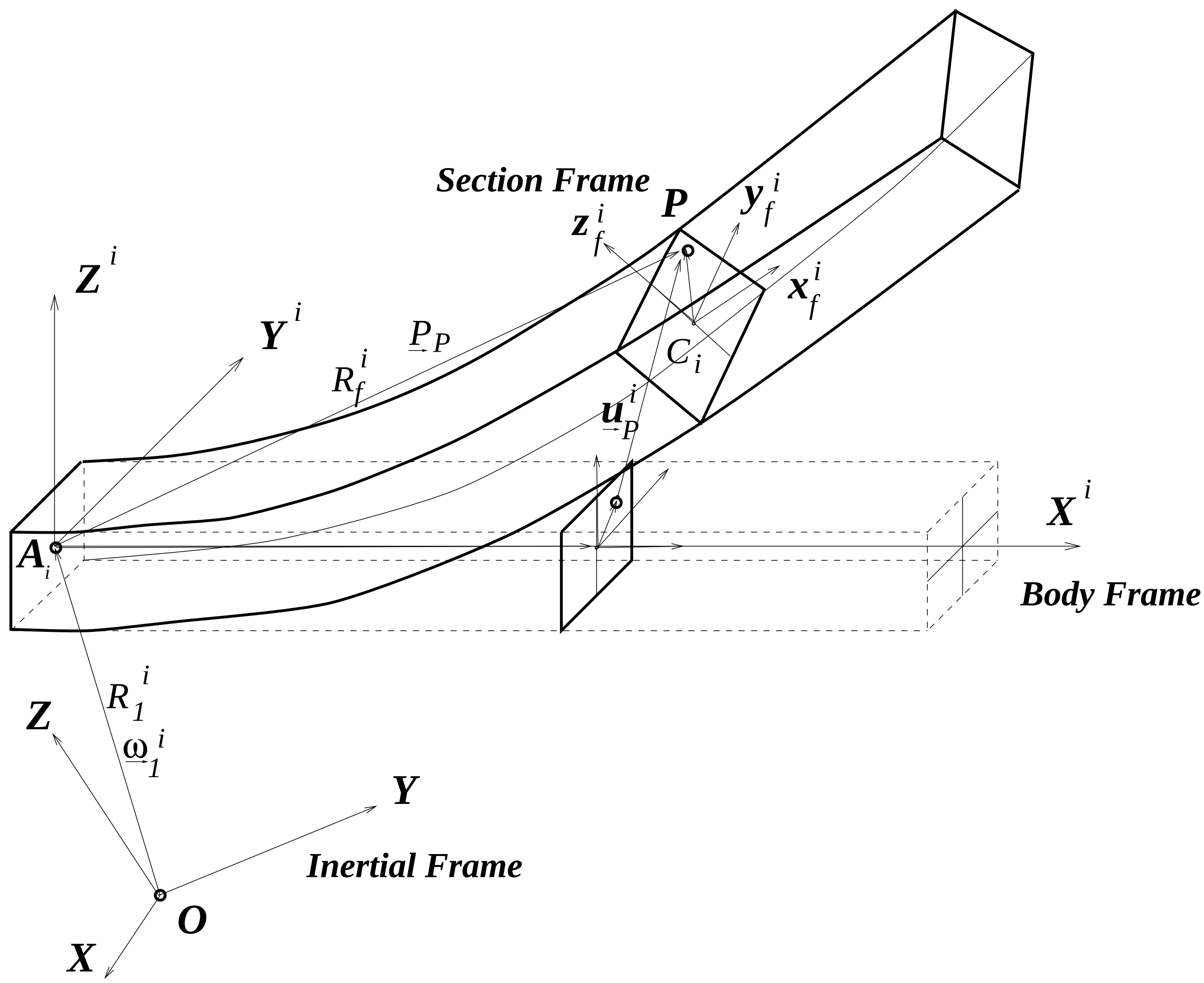
where $\mathbf{G}$ is easily shown to be an orthogonal complement to the Jacobian $\mathbf{\Phi}_{\mathbf{q}}$. Thus, pre-multiplying the augmented equations by $\mathbf{G}^T$ eliminates $\boldsymbol{\lambda}$:

$$\begin{aligned} \mathbf{G}^T\mathbf{M}\ddot{\mathbf{q}} &= \mathbf{G}^T\mathbf{F} \\ \widetilde{\mathbf{M}}\ddot{\mathbf{q}} &= \mathbf{G}^T\mathbf{F} = \mathbf{C}\boldsymbol{\tau} + \widetilde{\mathbf{F}} \end{aligned}$$

These f symbolic dynamic equations are very useful for inverse dynamic analyses, in which one solves for f actuator loads $\boldsymbol{\tau}$ that appear linearly.

In a forward dynamic analysis, the f dynamic equations can be solved simultaneously with the m algebraic constraint equations for $\mathbf{q}(t)$. Alternatively, $\mathbf{G}$ can be used to transform $\ddot{\mathbf{q}}$ into $\ddot{\mathbf{q}}_i$, with a $f \times f$ mass matrix $\mathbf{G}^T\mathbf{M}\mathbf{G}$.

- **Flexible beam model:**
  - Floating reference frame approach
  - Shear and warping neglected, inertia of cross-section included (Rayleigh)
  - Small strains (but possibly large deflections)
  - Elastic coordinates u, v, w, and ϕ represent the $O(2)$ displacement and rotation of any cross-section along the beam



To calculate strains, the $O(2)$ displacement field is used:

$$\mathbf{P}_P = \left\{ \begin{array}{l} x + u - \frac{1}{2}\int_0^x (v'^2 + w'^2) d\xi + (-v' + \underline{u'v' - w'\phi})y \\ +(-w' + \underline{u'w' + v'\phi})z \\ v + (1 \underline{- \frac{v'^2}{2} - \frac{\phi^2}{2}})y + (-\phi \underline{- \frac{v'w'}{2}})z \\ w + (\phi \underline{- \frac{v'w'}{2}})y + (1 \underline{- \frac{w'^2}{2} - \frac{\phi^2}{2}})z \end{array} \right\}$$

The $O(2)$ elastic rotation matrix is:

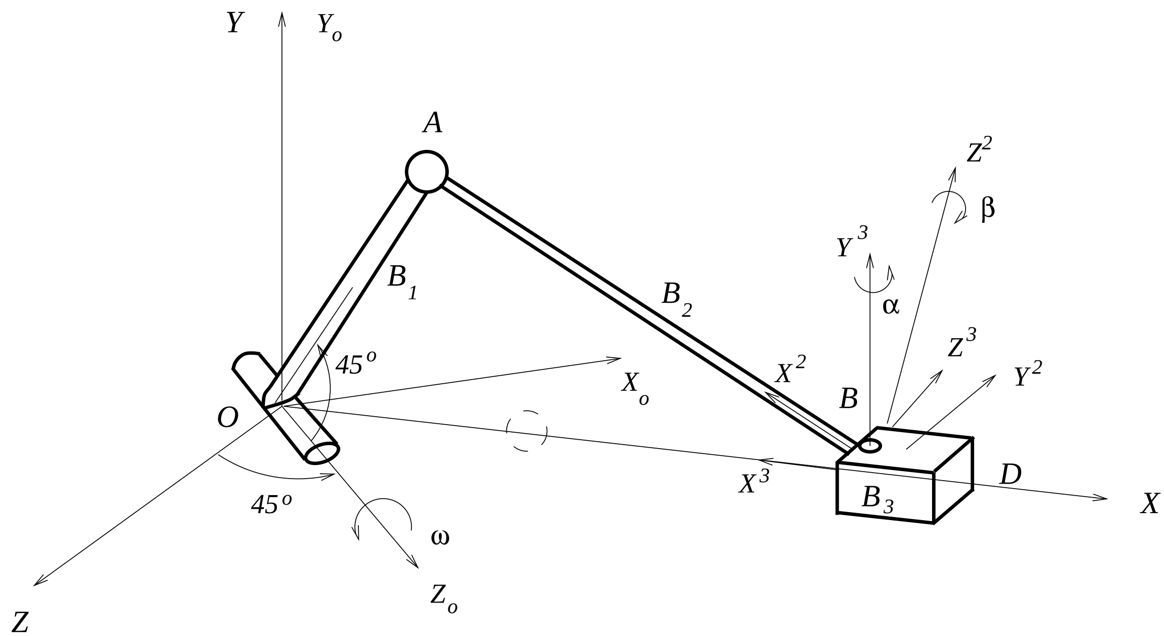
$$\mathbf{R}_f = \begin{bmatrix} 1 - \frac{1}{2}(v'^2 + w'^2) & -v' \underline{+ u'v' - w'\phi} & -w' \underline{+ u'w' + v'\phi} \\ v' - u'v' & 1 - \frac{v'^2}{2} - \frac{\phi^2}{2} & -\phi \underline{- \frac{v'w'}{2}} \\ w' - u'w' & \phi \underline{- \frac{v'w'}{2}} & 1 - \frac{w'^2}{2} - \frac{\phi^2}{2} \end{bmatrix}$$

The elastic coordinates are modelled using a Rayleigh-Ritz approach:

$$\begin{aligned} u(x,t) &= \sum_{i=1}^{a} x^i u_i(t) = x u_1 + x^2 u_2 + \cdots + x^a u_a \\ v(x,t) &= \sum_{i=1}^{b} x^{i+1} v_i(t) = x^2 v_1 + x^3 v_2 + \cdots + x^{b+1} v_b \\ w(x,t) &= \sum_{i=1}^{c} x^{i+1} w_i(t) = x^2 w_1 + x^3 w_2 + \cdots + x^{c+1} w_c \\ \phi(x,t) &= \sum_{i=1}^{d} x^i \phi_i(t) = x \phi_1 + x^2 \phi_2 + \cdots + x^d \phi_d \end{aligned}$$

where the shape functions are Taylor, Legendre, or Chebyshev polynomials.

Spatial flexible slider-crank mechanism [Jonker, 1989]



$O(2)$ (solid) and $O(1)$ (dotted) results for mid-point deflection on XOZ plane: